LEDAkem: a post-quantum key encapsulation mechanism based on QC-LDPC codes

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Code-based crypto

- Code-based public-key cryptosystems were introduced by McEliece in 1978, exploiting Goppa codes.
- Besides quantum resistant, they are algorithmically efficient.

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- Goppa codes have resisted cryptanalysis for 40 years...
- ...but they are large to store and slow to decode.

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Goppa code replacements (in the Hamming metric)

Goppa codes
[McEliece78]

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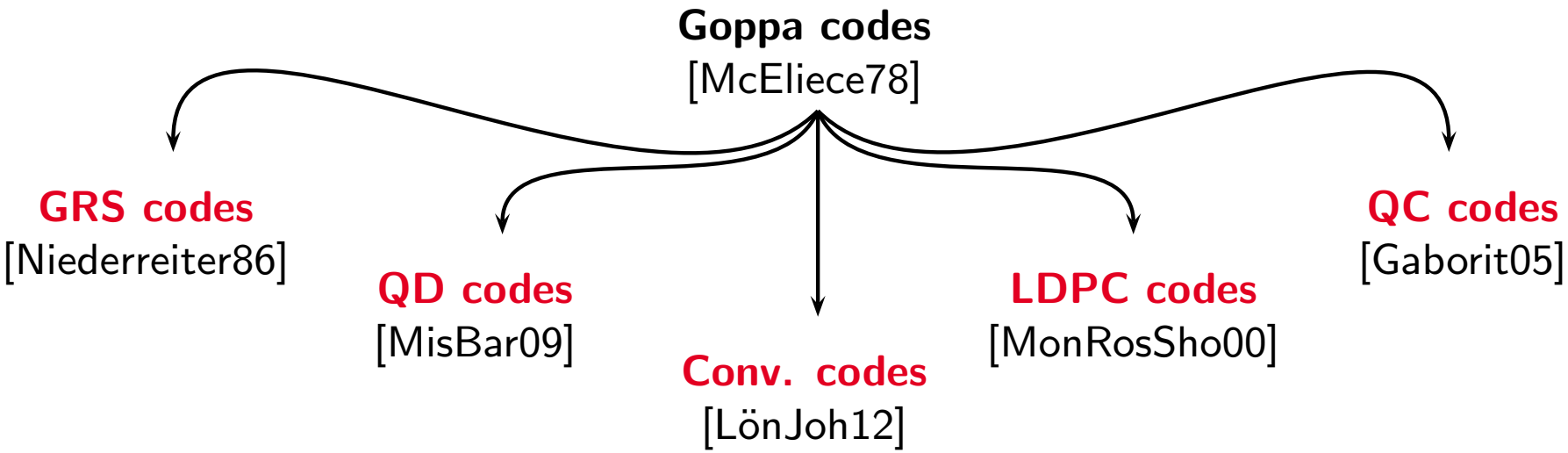

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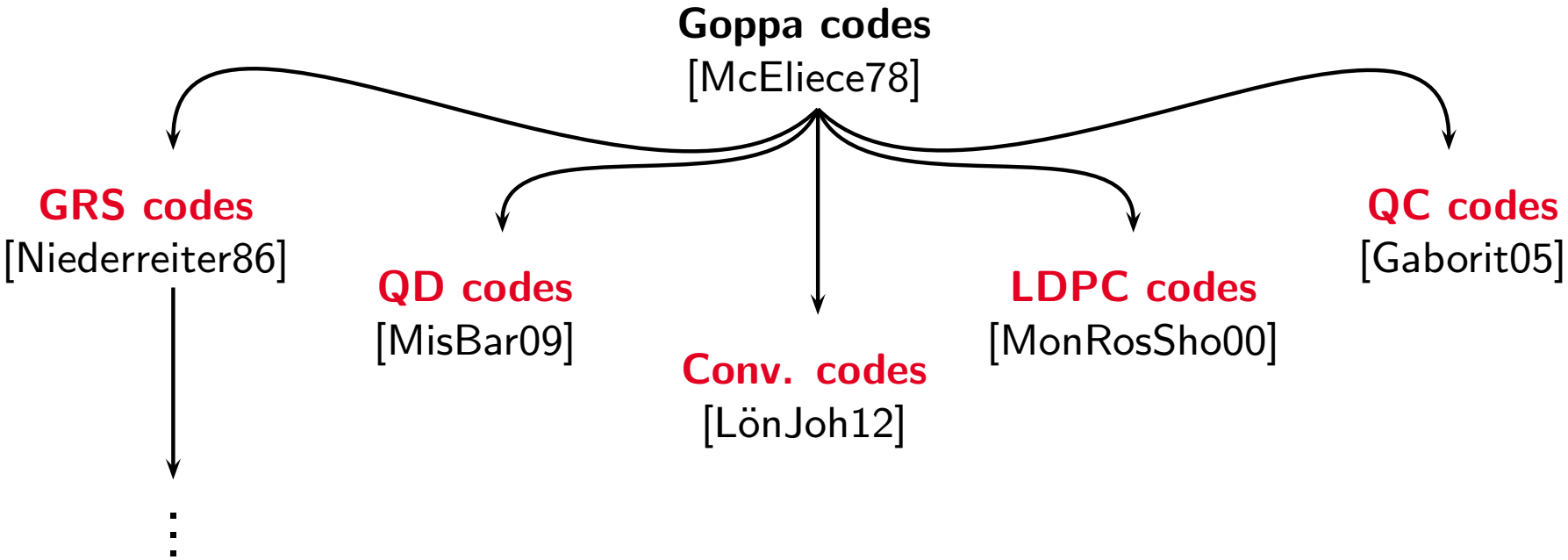

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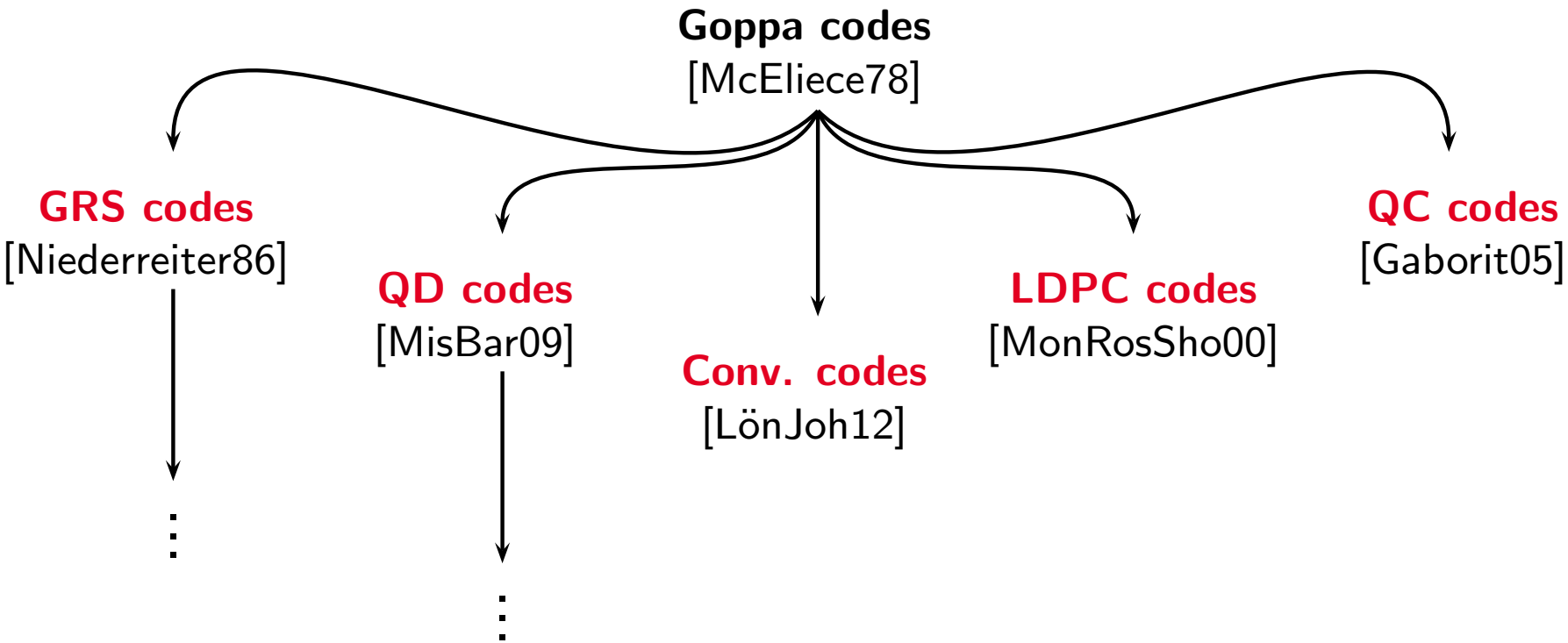

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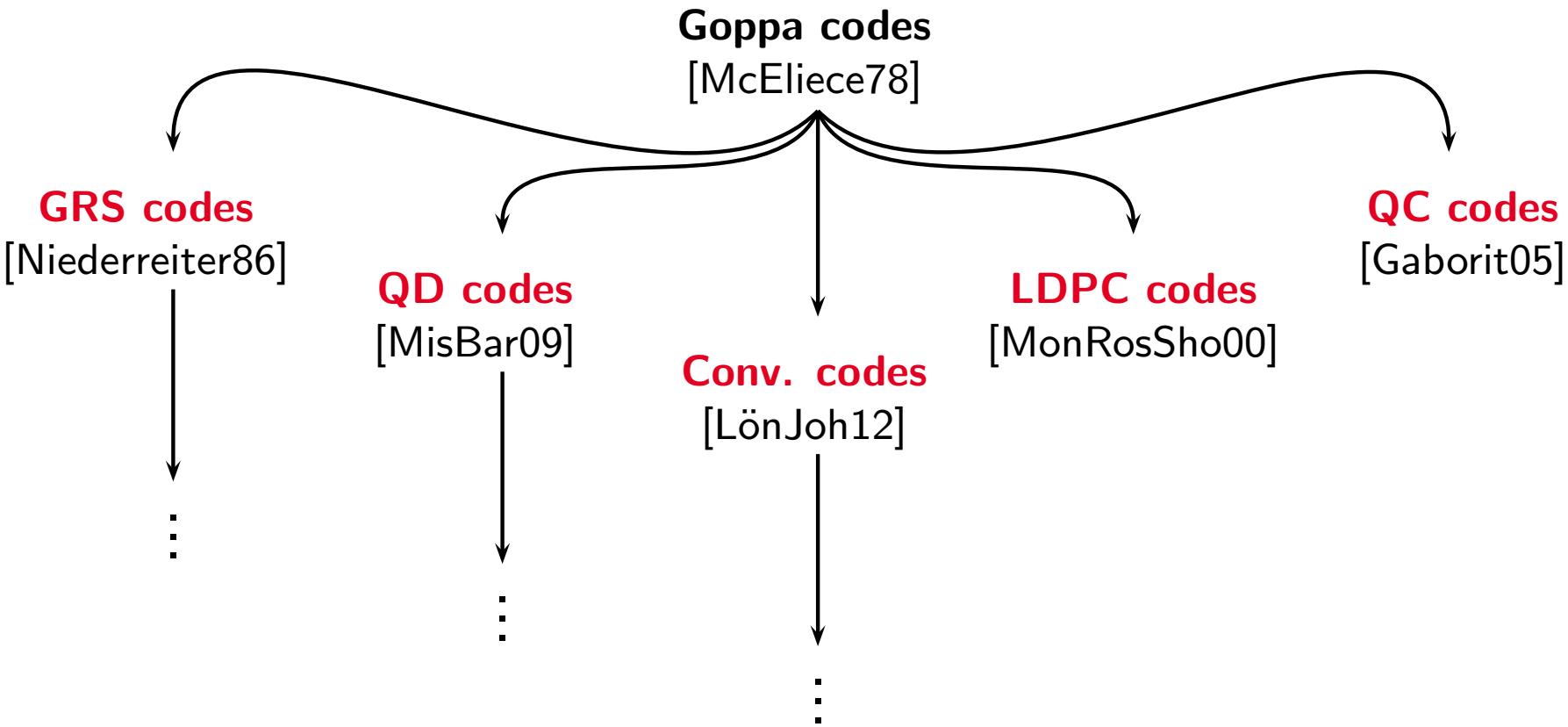

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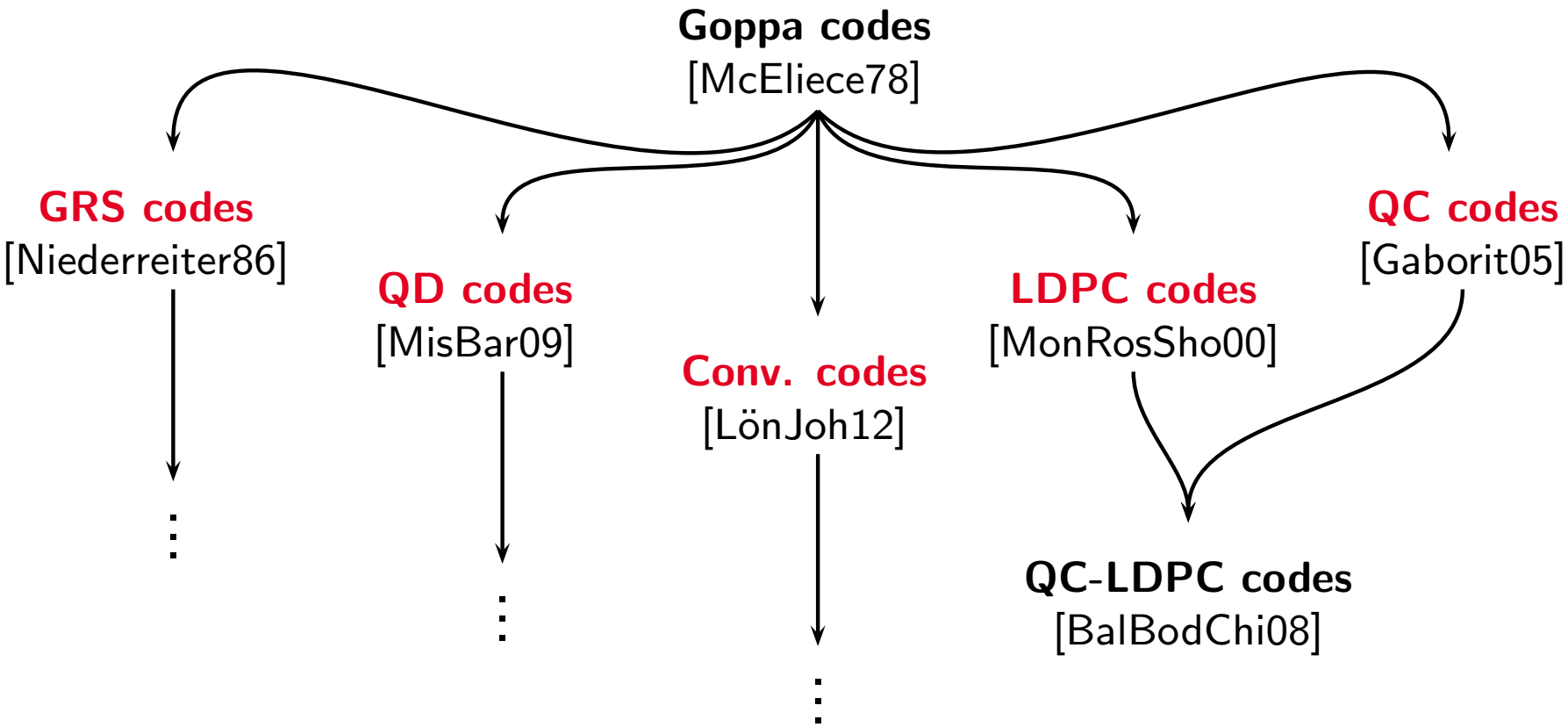

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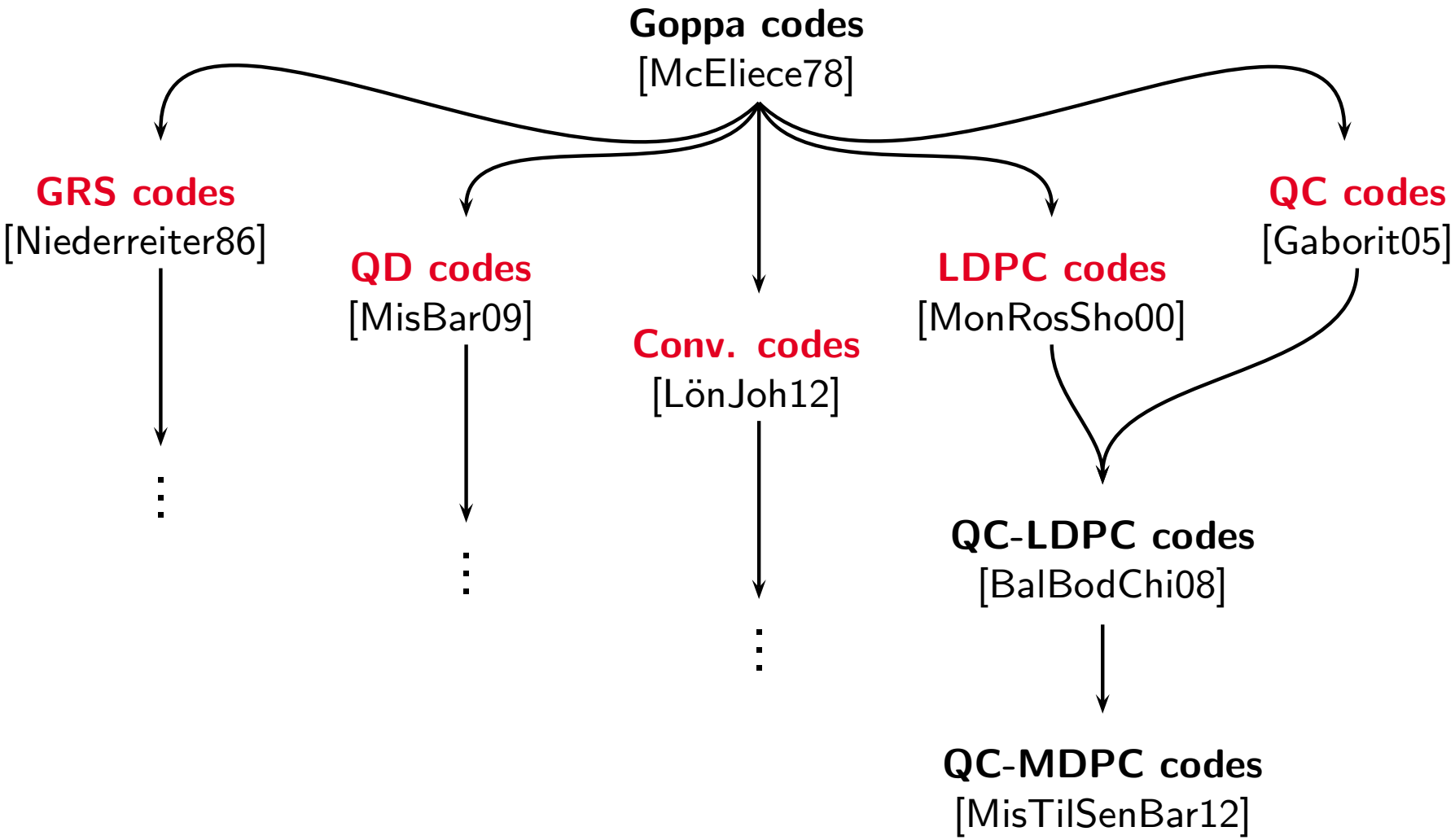

LDPC codes in the McEliece cryptosystem

- Low-density parity-check (LDPC) codes are capacity-achieving codes under belief propagation (BP) decoding.
- They allow a random-based design, which results in large key spaces.
- The low density of their matrices could be attractive to achieve compact representations.
- All these makes them interesting for the use in McEliece/Niederreiter.

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Warning

Public codes cannot be LDPC codes as well, otherwise secret codes are likely to be exposed.

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- M. Baldi, F. Chiaraluce, "Cryptanalysis of a new instance of McEliece cryptosystem based on QC-LDPC codes," Proc. IEEE ISIT 2007, Nice, France, Jun. 2007, pp. 2591–2595.
- A. Otmani, J.P. Tillich, L. Dallot, "Cryptanalysis of two McEliece cryptosystems based on quasi-cyclic codes," Proc. SCC 2008, Beijing, China, Apr. 2008.

LDPC codes in the McEliece cryptosystem (2)

Solutions

1. Use a transformation matrix $\mathbf{Q}$ that hides the sparse parity-check matrix $\mathbf{H}$ of the private code into a denser parity-check matrix $\mathbf{L} = \mathbf{H} \cdot \mathbf{Q}$ of the public code [BBC2008].
2. Use a private code defined by a denser parity-check matrix, called moderate-density parity-check (MDPC) matrix, and a permutation-equivalent public code [MTSB2013].

Pros and cons of 1 :

- Iterative decoding is more efficient on more sparse matrices.
- Multiplication by $\mathbf{Q}$ increases the weight of error vectors to be corrected during decryption.

- M. Baldi, M. Bodrato, F. Chiaraluce, "A new analysis of the McEliece cryptosystem based on QC-LDPC codes", Proc. SCN 2008, vol. 5229 of LNCS, pp. 246–262, 2008.
- R. Misoczki, J.-P. Tillich, N. Sendrier, P.S.L.M. Barreto, "MDPC-McEliece: new McEliece variants from moderate density parity-check codes", Proc. IEEE ISIT 2013, pp. 2069–2073, July 2013.

Example of QC-(almost)LDPC code

$$\mathbf{H} = \left[\begin{array}{ccccccccccc|ccccccccccc}
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
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0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1
\end{array}\right]$$

- Number of ciculant blocks: $n_0 = 2$.
- Code rate: $R = \frac{n_0 - 1}{n_0} = 1/2$.
- Parity-check matrix column weight: $d_v = 3$.
- Parity-check matrix row weight: $d_c = n_0 d_v = 6$.

QC-LDPC and QC-MDPC codes

$$\mathbf{H} = \left[\begin{array}{cccccccccc|cccccccccc} \mathbf{1} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & & & & & & \vdots & & & & & & \end{array}\right]$$

- The parity-check matrix is described by its first row.
- The storage size increases linearly in the code length.
- The code length is usually very large ($10,000 \lessapprox n \lessapprox 100,000$)
- $\mathbf{H}$ defines a QC-LDPC code if d_v is $\approx \log(n)$ ($d_v \lessapprox 20$).
- $\mathbf{H}$ defines a QC-MDPC code if d_v is $\approx \sqrt{n}$ ($50 \lessapprox d_v \lessapprox 150$).

Bit flipping decoding

- Classical hard-decision iterative decoding algorithms for LDPC codes are known as bit flipping (BF) algorithms.
- They are well suited when soft information from the channel is not available (as in the McEliece cryptosystem).
- From [Gallager1962]:

> The decoder computes all the parity checks and then changes any digit that is contained in more than some fixed number of unsatisfied parity-check equations. Using these new values, the parity checks are recomputed, and the process is repeated until the parity checks are all satisfied.

▶ R. G. Gallager, "Low-density parity-check codes," IRE Trans. Inform. Theory, vol. 8, pp. 21–28, 1962.

Contribution

1

Propose a *Low dEnsity coDe-bAsed key encapsulation mechanism* (LEDAkem).

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Propose an ad-hoc decoding algorithm exploiting the knowledge of $\mathbf{Q}$ to achieve better performance.

3

Propose a reference and portable C99 implementation of LEDAkem.

LEDAkem functions

Key Generation

1. Generate a random $r \times n$ binary block circulant matrix $\mathbf{H} = [\mathbf{H}_0, \ldots, \mathbf{H}_{n_0-1}]$ with column weight $d_v \ll n$
2. Generate a random, non-singular, $n \times n$ binary block circulant matrix $\mathbf{Q}$ with column weight $m \ll n$
3. Compute $\mathbf{L} = \mathbf{H} \cdot \mathbf{Q} = [\mathbf{L}_0, \ldots, \mathbf{L}_{n_0-1}]$
4. Private key: $\mathbf{H}, \mathbf{Q}$; Public Key: $\mathbf{M} = (\mathbf{L}_{n_0-1})^{-1} \cdot \mathbf{L}$

LEDAkem functions (2)

Key Encapsulation

1. Generate a random n-bit error vector $\mathbf{e}$ with weight t
2. Compute the ciphertext (syndrome) $\mathbf{s} = \mathbf{M}\mathbf{e}^T$
3. Derive the shared secret $\mathbf{x} = \mathrm{KDF}(\mathbf{e})$

Key Decapsulation

1. Obtain $\mathbf{e}$ as $\textsc{Decode}(\mathbf{s}, \mathbf{H}, \mathbf{Q})$
2. Derive the shared secret $\mathbf{x} = \mathrm{KDF}(\mathbf{e})$

General attacks against McEliece/Niederreiter

Decoding attacks

Aimed at decrypting one or more ciphertexts without knowing the private key.

Key recovery attacks

Aimed at recovering the private key from the public key.

Decoding attacks

- The most dangerous decoding attacks (DAs) exploit information set decoding (ISD).
- The ISD principle was introduced by Prange in 1962.
- The first efficient algorithms were introduced by Lee-Brickell and Leon-Stern in 1988/89.
- These techniques have known great advancements in recent years.

- E. Prange, "The use of information sets in decoding cyclic codes, Information Theory," IRE Transactions on, vol. 8, no. 5, pp. 5–9, 1962.
- P. Lee, E. Brickell, "An observation on the security of McElieces public-key cryptosystem," Advances in Cryptology - EUROCRYPT 88, pp 275–280, 1988.
- J. Leon, "A probabilistic algorithm for computing minimum weights of large error-correcting codes," IEEE Trans. Inform. Theory, vol. 34, no. 5, pp. 1354–1359, 1988.
- J. Stern, "A method for finding codewords of small weight," Coding Theory and Applications, vol. 388 of Springer LNCS, pp. 106-113, 1989.

Modern information set decoding

- The general decoding problem can be reduced to that of searching low weight codewords.
- Modern approaches exploit the birthday paradox to search for low weight codewords.
- Lower bounds on complexity have been found recently by Niebuhr et al.

- C. Peters, "Information-set decoding for linear codes over F_q," Post-Quantum Cryptography, vol. 6061 of Springer LNCS, pp. 81–94, 2010.
- D. J. Bernstein, T. Lange, C. Peters, "Smaller decoding exponents: ball-collision decoding," CRYPTO 2011, vol. 6841 of Springer LNCS, pp 743–760, 2011.
- A. May, A. Meurer, E. Thomae, "Decoding random linear codes in $O(2^{0.054n})$," ASIACRYPT 2011, vol. 7073 of Springer LNCS, pp. 107124, 2011.
- A. Becker, A. Joux, A. May, and A. Meurer, "Decoding random binary linear codes in $2^{n/20}$: How $1 + 1 = 0$ improves information set decoding," Advances in Cryptology - EUROCRYPT 2012, vol. 7237 of Springer LNCS, pp. 520–536, 2012.
- R. Niebuhr, E. Persichetti, P.-L. Cayrel, S. Bulygin, J. Buchmann, "On lower bounds for information set decoding over F_q and on the effect of partial knowledge," Int. J. Inf. Coding Theory, vol. 4, no. 1, pp. 47–78, 2017.

Pre-quantum VS post-quantum decoding attacks

- Grover's algorithm is a quantum algorithm introduced for finding all the roots of a Boolean function $\{0,1\}^n \rightarrow \{0,1\}$ in $O(\sqrt{2^n})$ instead of $O(2^n)$.

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- For searching one entry of an unsorted list of n entries,
  - The best classical algorithm requires $n/2$ steps on average.
  - Grover's algorithm requires $\pi/4\sqrt{n}$ steps using $\log_2(n)$ qubits.

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- Grover's algorithm reduces the number of iterations but does not reduce the cost per iteration.
- However, it somehow impacts the work factor of ISD.

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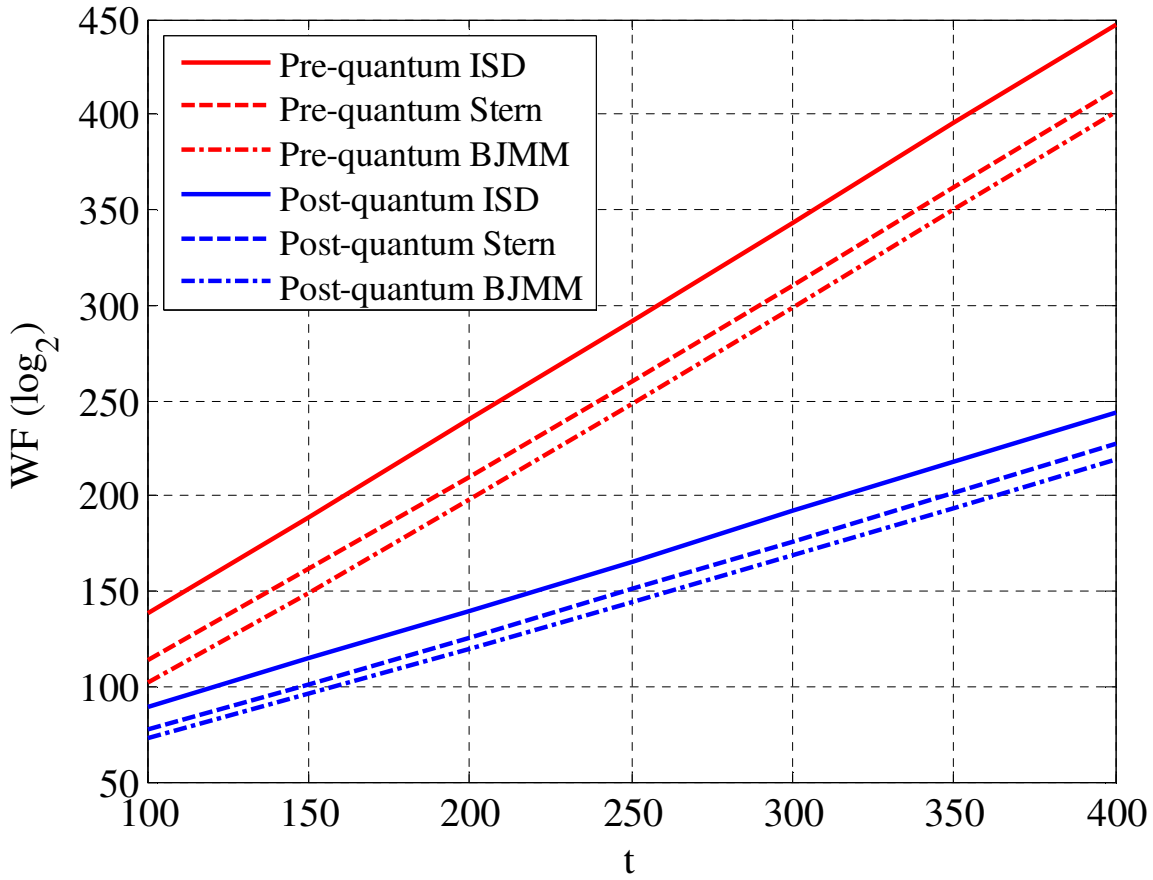
Pre-quantum VS post-quantum decoding attacks

Pre- and post-quantum WF of some ISD algorithms versus t, for codes with $n = 12000$, $k = 6000$.



Key recovery attacks based on ISD

- The matrix $\mathbf{L}$, with row weight $\leq n_0 m d_v$, is a valid parity-check matrix for the public code.
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Work Factor of ISD-based attacks

$$\mathrm{WF}_{\mathrm{DA}} = \frac{\mathrm{C}_{\mathrm{ISD}}(n, k, t)}{\sqrt{p}}, \quad \mathrm{WF}_{\mathrm{KRA}} = \frac{\mathrm{C}_{\mathrm{ISD}}(n, n-k, n_0 m d_v)}{p}$$

▶ N. Sendrier, "Decoding one out of many," in Proc. PQCrypto 2011, vol. 7071 of Springer LNCS, pp. 51–67, 2011.

Key recovery attacks based on decoding errors

- Recently, it has been shown that QC-MDPC and QC-LDPC code-based McEliece cryptosystem may suffer from reaction attacks exploiting decoding errors.

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- The attack is built upon two facts:
  1. The decryption failure probability is non-zero and depends on the structure of the secret key.
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- LEDAkem thwarts these attacks by using ephemeral (i.e., one-time) keypairs.

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Rationale of the Q-decoder

- Decoding is performed on the syndrome

$$\mathbf{s} = \mathbf{e} \cdot \mathbf{L}^T = \mathbf{e} \cdot \mathbf{Q}^T \cdot \mathbf{H}^T = \mathbf{e}' \cdot \mathbf{H}^T$$

  where $\mathbf{e}' = \mathbf{e} \cdot \mathbf{Q}^T$ is the expanded error vector.
- Let $\phi(\mathbf{e})$ denote the support of $\mathbf{e}$ and $\mathbf{q}_j$ be the j-th row of $\mathbf{Q}^T$, then

$$\mathbf{e}' = \sum_{j \in \phi(\mathbf{e})} \mathbf{q}_j$$

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- Also $\mathbf{e}$ is sparse ($\mathrm{wt}(\mathbf{e}) = t \ll n$), hence

$$\mathrm{wt}(\mathbf{e}') \approx mt$$

Rationale of the Q-decoder (2)

- Let us consider the (integer) inner product $\rho = \mathbf{e}' * \mathbf{q}_v$:
  - if $v \notin \phi(\mathbf{e})$, then the supports of $\mathbf{e}'$ and $\mathbf{q}_v$ have a small intersection and ρ is small;
  - if $v \in \phi(\mathbf{e})$, then $\mathbf{q}_v$ is one of the rows forming $\mathbf{e}'$, hence ρ is large.

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- As in BF decoding, an estimate of $\mathbf{e}'$ is obtained by computing the (integer) inner product between the syndrome and each column of $\mathbf{H}$

$$\mathbf{\Sigma} = \mathbf{s} * \mathbf{H}$$

and thresholding the vector $\mathbf{\Sigma}$.

Rationale of the Q-decoder (2)

- Let us consider the (integer) inner product $\rho = \mathbf{e}' * \mathbf{q}_v$:
  - if $v \notin \phi(\mathbf{e})$, then the supports of $\mathbf{e}'$ and $\mathbf{q}_v$ have a small intersection and ρ is small;
  - if $v \in \phi(\mathbf{e})$, then $\mathbf{q}_v$ is one of the rows forming $\mathbf{e}'$, hence ρ is large.
- As in BF decoding, an estimate of $\mathbf{e}'$ is obtained by computing the (integer) inner product between the syndrome and each column of $\mathbf{H}$

$$\mathbf{\Sigma} = \mathbf{s} * \mathbf{H}$$

  and thresholding the vector $\mathbf{\Sigma}$.
- So we can estimate $\phi(\mathbf{e})$ by replacing $\mathbf{e}'$ with $\mathbf{\Sigma}$ to compute

$$\mathbf{R} = [\rho_0, \rho_1, \cdots, \rho_{n-1}] = \mathbf{\Sigma} * \mathbf{Q}$$

  and thresholding the vector $\mathbf{R}$.

Q-decoder

Initialization

$\mathbf{e} = \mathbf{0}_{1\times n}, \quad \mathbf{s} = \mathbf{e}\cdot\mathbf{L}^T.$

Description of the j-th iteration

Input: $\mathbf{e}^{(j-1)}, \mathbf{s}^{(j-1)}$

1. Compute $\mathbf{\Sigma} = [\sigma_0, \sigma_1, \cdots, \sigma_{n-1}] = \mathbf{s}^{(j-1)} * \mathbf{H}$.
2. Compute $\mathbf{R} = [\rho_0, \rho_1, \cdots, \rho_{n-1}] = \mathbf{\Sigma} * \mathbf{Q}$.
3. Compute $\Psi = \left\{ i \,\middle|\, \rho_i \geq b^{(j)} \right\}$.
4. Update the error vector as $\mathbf{e}^{(j)} = \mathbf{e}^{(j-1)} + \mathbf{1}_\Psi$.
5. Update the syndrome as $\mathbf{s}^{(j)} = \mathbf{s}^{(j-1)} + \sum_{i\in\Psi} \mathbf{q}_i \cdot \mathbf{H}^T$.

Output: $\mathbf{e}^{(j)}, \mathbf{s}^{(j)}$

Choice of the flipping thresholds

- Through combinatorial arguments (details in the paper) we can estimate $P\{e_i = 1 | \rho_i\}$.

- J. Chaulet and N. Sendrier, "Worst case QC-MDPC decoder for McEliece cryptosystem," Proc. IEEE ISIT 2016, Barcelona, 2016, pp. 1366–1370.

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- Let us define a decision margin $\Delta \geq 0$ and consider the decision condition

$$P\{e_i = 1|\rho_i\} > (1+\Delta)P\{e_i = 0|\rho_i\} = \frac{1+\Delta}{2+\Delta},$$

since $P\left\{e_i = 0|\rho_i^{(l)}\right\} = 1 - P\left\{e_i = 1|\rho_i^{(l)}\right\}$.

- Increasing Δ increases the average number of iterations, but reduces the DFR.

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- Increasing Δ increases the average number of iterations, but reduces the DFR.
- The corresponding decision threshold value is

$$b = \min\left\{\rho_i \in [0; md_v], \text{ s.t. } P\{e_i = 1|\rho_i\} > \frac{1+\Delta}{2+\Delta}\right\}$$

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Choice of the flipping thresholds (2)

- $P\{e_i = 1|\rho_i\}$ depends on the weight of the error vector.
- The weight of the error vector can be related to the syndrome weight (w).
- We can exploit a look-up table populated with the pairs $\{w, b\}$, sequentially ordered.
- During the l-th iteration:
  1. compute the syndrome weight $w^{(l)}$,
  2. search the largest w in the look-up table such that $w < w^{(l)}$,
  3. set $b^{(l)} = b$.

Proposed parameters

Table: Parameters for LEDAkem and estimated computational efforts to break a given instance as a function of the security category and number of circulant blocks n_0

Category	n_0	p	d_v	$[m_0, \cdots, m_{n_0-1}]$	t	$SL_{DA}^{(pq)}$	$SL_{KRA}^{(pq)}$	$SL_{DA}^{(cl)}$	$SL_{KRA}^{(cl)}$	DFR
1	2	27,779	17	[4, 3]	224	135.43	134.84	217.45	223.66	$\approx 8.3 \cdot 10^{-9}$
	3	18,701	19	[3, 2, 2]	141	135.63	133.06	216.42	219.84	$\lesssim 10^{-9}$
	4	17,027	21	[4, 1, 1, 1]	112	136.11	139.29	216.86	230.61	$\lesssim 10^{-9}$
2–3	2	57,557	17	[6, 5]	349	200.47	204.84	341.52	358.16	$\lesssim 10^{-8}$
	3	41,507	19	[3, 4, 4]	220	200.44	200.95	341.61	351.57	$\lesssim 10^{-8}$
	4	35,027	17	[4, 3, 3, 3]	175	200.41	201.40	343.36	351.96	$\lesssim 10^{-8}$
4–5	2	99,053	19	[7, 6]	474	265.38	267.00	467.24	478.67	$\lesssim 10^{-8}$
	3	72,019	19	[7, 4, 4]	301	265.70	270.18	471.67	484.48	$\lesssim 10^{-8}$
	4	60,509	23	[4, 3, 3, 3]	239	265.48	268.03	473.38	480.73	$\lesssim 10^{-8}$

- Public key size $= (n_0 - 1)p$.
- Ranging between 3 and 22 KiB.

Efficient implementation

Circulant matrix representation/arithmetics

- Represent circulant blocks as elements of $\mathbb{F}_2[x]/\langle x^p + 1\rangle$
  - Reduces both time and space complexity for arithmetics

Remove invertibility check for $\mathbf{Q}$

- Perm($\mathbf{Q}$) is odd and $< p \Rightarrow \mathbf{Q}$ is invertible

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- Reference implementation in ISO-C99 without platform-specific optimization.
  - NIST ref. platform: Base Intel x86-64 ISA (early 2006 CPUs).
- Possible further optimizations:
  - Sub-quadratic poly multiplication arithmetics.
  - Use x86-64 ISA extensions (e.g. CLMUL) and vector units.
  - Design a dedicated HW implementation.

Implementation results

Table: Running times for key generation, encryption and decryption as a function of the category and the number of circulant blocks n_0 on an AMD Ryzen 5 1600 CPU.

Category	n_0	KeyGen (ms)	Encrypt (ms)	Decrypt (ms)	Total CPU time Ephemeral KEM (ms)
1	2	34.11 ($\pm$1.07)	2.11 ($\pm$0.08)	16.78 ($\pm$0.53)	52.99
	3	16.02 ($\pm$0.26)	2.15 ($\pm$0.17)	21.65 ($\pm$1.71)	39.81
	4	13.41 ($\pm$0.23)	2.42 ($\pm$0.08)	24.31 ($\pm$0.86)	40.14
2–3	2	142.71 ($\pm$1.52)	8.11 ($\pm$0.21)	48.23 ($\pm$2.93)	199.05
	3	76.74 ($\pm$0.78)	8.79 ($\pm$0.20)	49.15 ($\pm$2.20)	134.68
	4	54.93 ($\pm$0.84)	9.46 ($\pm$0.28)	46.16 ($\pm$2.03)	110.55
4–5	2	427.38 ($\pm$5.15)	23.00 ($\pm$0.33)	91.78 ($\pm$5.38)	542.16
	3	227.71 ($\pm$1.71)	24.85 ($\pm$0.37)	92.42 ($\pm$4.50)	344.99
	4	162.34 ($\pm$2.39)	26.30 ($\pm$0.53)	127.16 ($\pm$4.42)	315.80

Thanks for the attention

Questions?

https://www.ledacrypt.org