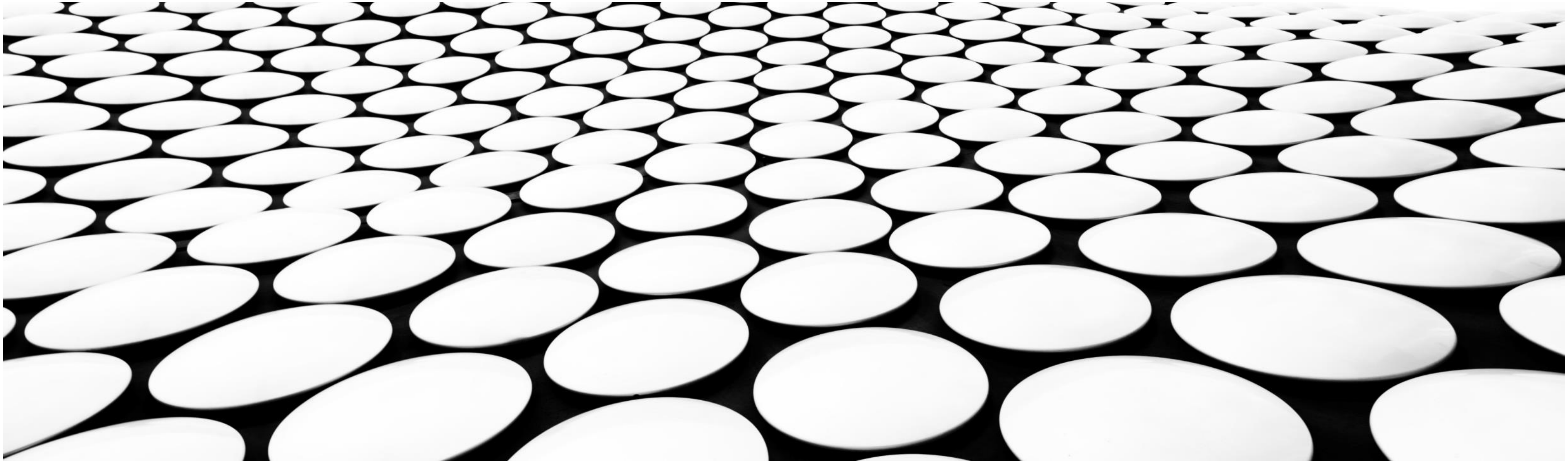
MATH CIRCLE AT FAU

MORE COUNTING, THIS AND THAT



A QUICK REFRESHER

- Ms. Nakamura's class has 25 students. Ms. Nakamura wants to select 7 students to represent the class at the next festivities. In how many different ways can she do this?

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n! &= 1 \times 2 \times 3 \times \cdots \times n \\
1! &= 1,\ 2! = 2,\ 3! = 6,\ 4! = 24,\ 5! = 120,\ \ldots \\
0! &= 1 \ \text{(Because it works)} \\
\binom{n}{k} &= \frac{n(n-1)\cdots(n-k+1)}{k!}, \\
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\binom{n}{3} &= \frac{n(n-1)(n-2)}{6}, \ldots, \binom{n}{n} = 1. \\
\binom{n}{k} &= 0 \text{ if } k > n.
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$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
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Solution: $\binom{25}{7} = 480{,}700$

MS. NAKAMURA’S CLASS AGAIN

Ms. Nakamura’s class has 25 students, 10 boys and 15 girls. Of the students, 4 boys and 7 girls are excellent singers; the rest of the students are just so-so. Ms. Nakamura has to assemble a cast for a production of an opera. She needs 2 boys and 3 girls with excellent voices for the lead roles, and then a chorus of 5 boys and 5 girls from among the remaining students, making sure that the excellent singers not chosen for the lead roles are part of the chorus. In how many different ways can such a cast be assembled?

MS. NAKAMURA'S CLASS AGAIN, SOLVED

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Solution. Ms. Nakamura has to choose 2 boys from 4, 3 girls from 7, for the lead roles. For the chorus, since the two remaining boys and the 4 remaining girls are automatically in it, she has to select 3 boys from 6 and 2 girls from 11. The total number of different ways this can be done is

$$\binom{4}{2} \times \binom{7}{3} \times \binom{6}{3} \times \binom{11}{2} = 6 \times 35 \times 20 \times 55 = 231000 \text{ different ways.}$$

NOW FOR SOMETHING COMPLETELY DIFFERENT

- Two people left at dawn, at the exact same time, one traveling from A to B, the other one from B to A. They travel at a constant speed, without stopping. They meet at noon. The first one arrives at B at 4 p.m., the second one arrives at A at 9 p.m.

- At what time was dawn that day?

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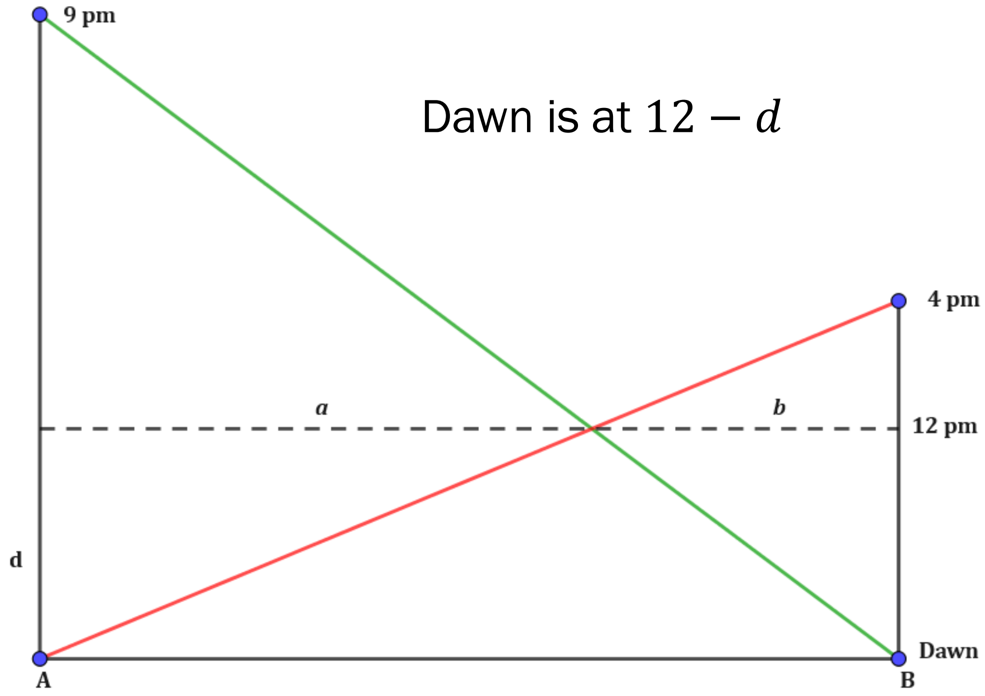
$\frac{d}{4} = \frac{a}{b}, \frac{d}{9} = \frac{b}{a}$

Since $\frac{a}{b} = \frac{b}{a}$ we get $\frac{d}{4} = \frac{9}{d}$ thus $d^2 = 36.$

So $d = 6, 12 - d = 6.$

Dawn is at 6 am.

Dawn is at $12 - d$



THE 2, 3, 5 QUESTION

- How many numbers in the range 1-1000 are NOT divisible by 2, 3, or 5?

THE 2, 3, 5 QUESTION SOLVED

- How many numbers in the range 1-1000 are NOT divisible by 2, 3, or 5?
- It is easier to count numbers that are multiples of 2, 3, or 5.
- Principle of ***Inclusion-Exclusion:***
- #(Multiples of 2, 3, 5) = #(Mult. 2) + #(Mult. 3) + #(Mult.5) - #(Mult.6) - #(Mult, 10) - #(Mult, 15) + #(Mult 30).
- Works out to $500 + 333 + 200 - 166 - 100 - 66 + 33 = 734$
- There are 1000 numbers in the range, $1000 - 734 = 266$.
- The answer is 266.

A TILING PROBLEM

- Leonardo wishes to tile a rectangular path that is 10 feet 120 inches long and 10inches wide. He wants to use 24 $10'' \times 5''$ rectangular tiles, all of the same green color, to do this. In how many different ways can Leonardolay the tiles?

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- He could start this way:

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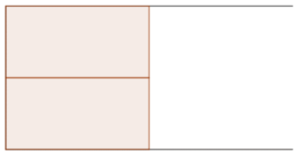

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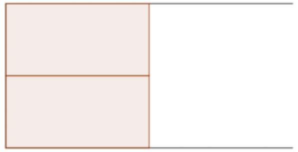

Once you figure out the answer, here are two additional challenges: What is the answer if the strip to be tiled is $15''$ wide? What if the tiles are of different colors? Say black and white. (To think about at home)

TILING SOLUTION

- Suppose s_n is the number of different ways we can tile $5n$ inches of the strip. We need to find s_{24}. We saw $s_1 = 1, s_2 = 2$. How can we get to tile $5n$ inches, $n \geq 3$? We can either cover $5(n-1)$ inches and lay one vertical tile, or we can tile $5(n-2)$ inches and add two horizontal tiles. It follows that $s_n = s_{n-1} + s_{n-2}$. So here is how the values of s_n evolve:
- 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025,
- There are 75,025 different ways to tile the strip.