MATH CIRCLE AT FAU

MORE COUNTING, THIS AND THAT
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\[ n! = 1 \times 2 \times 3 \times \cdots \times n \]
\[ 1! = 1, \ 2! = 2, \ 3! = 6, \ 4! = 24, \ 5! = 120, \ldots \]
\[ 0! = 1 \quad (\text{Because it works}) \]

\[
\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!},
\]

\[
\binom{n}{0} = 1, \quad \binom{n}{1} = n, \quad \binom{n}{2} = \frac{n(n-1)}{2},
\]

\[
\binom{n}{3} = \frac{n(n-1)(n-2)}{6}, \ldots, \quad \binom{n}{n} = 1.
\]

\[
\binom{n}{k} = 0 \quad \text{if} \quad k > n.
\]
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\[
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\end{array}
\]

\[
\binom{25}{7} = 480,700
\]

Solution: \( \binom{25}{7} = 480,700 \)
Ms. Nakamura’s class has 25 students, 10 boys and 15 girls. Of the students, 4 boys and 7 girls are excellent singers; the rest of the students are just so-so. Ms. Nakamura has to assemble a cast for a production of an opera. She needs 2 boys and 3 girls with excellent voices for the lead roles, and then a chorus of 5 boys and 5 girls from among the remaining students, making sure that the excellent singers not chosen for the lead roles are part of the chorus. In how many different ways can such a cast be assembled?
MS. NAKAMURA’S CLASS AGAIN, SOLVED

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Solution. Ms. Nakamura has to choose 2 boys from 4, 3 girls from 7, for the lead roles. For the chorus, since the two remaining boys and the 4 remaining girls are automatically in it, she has to select 3 boys from 6 and 2 girls from 11. The total number of different ways this can be done is

\[
\binom{4}{2} \times \binom{7}{3} \times \binom{6}{3} \times \binom{11}{2} = 6 \times 35 \times 20 \times 55 = 231000 \text{ different ways.}
\]
NOW FOR SOMETHING COMPLETELY DIFFERENT

- Two people left at dawn, at the exact same time, one traveling from A to B, the other one from B to A. They travel at a constant speed, without stopping. They meet at noon. The first one arrives at B at 4 p.m., the second one arrives at A at 9 p.m.

- At what time was dawn that day?
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\[
\frac{d}{4} = \frac{a}{b}, \quad \frac{d}{9} = \frac{b}{a}
\]

Since \( \frac{a}{b} = \frac{b}{a} \) we get \( \frac{d}{4} = \frac{9}{d} \) thus \( d^2 = 36 \).

So \( d = 6, 12 - d = 6 \).

Dawn is at 6 am.
THE 2, 3, 5 QUESTION

- How many numbers in the range 1-1000 are NOT divisible by 2, 3, or 5?
THE 2, 3, 5 QUESTION SOLVED

- How many numbers in the range 1-1000 are NOT divisible by 2, 3, or 5?
- It is easier to count numbers that are multiples of 2, 3, or 5.
- Principle of Inclusion-Exclusion:
  - \( \#(\text{Multiples of 2, 3, 5}) = \#(\text{Mult. 2}) + \#(\text{Mult. 3}) + \#(\text{Mult. 5}) - \#(\text{Mult. 6}) - \#(\text{Mult. 10}) - \#(\text{Mult. 15}) + \#(\text{Mult 30}). \)
  - Works out to \( 500 + 333 + 200 - 166 - 100 - 66 + 33 = 734 \)
  - There are 1000 numbers in the range, \( 1000 - 734 = 266 \).
- The answer is 266.
Leonardo wishes to tile a rectangular path that is 10 feet 120 inches long and 10 inches wide. He wants to use 24 10’’ × 5’’ rectangular tiles, all of the same green color, to do this. In how many different ways can Leonardo lay the tiles?
A TILING PROBLEM

- Leonardo wishes to tile a rectangular path that is 10 feet = 120 inches long and 10 inches wide. He wants to use 24 10 ″ × 5 ″ rectangular tiles, all of the same color, to do this. In how many different ways can Leonardo lay the tiles?
- He could start this way:
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Once you figure out the answer, here are two additional challenges: What is the answer if the strip to be tiled is 15” wide? What if the tiles are of different colors? Say black and white. (To think about at home)
TILING SOLUTION

- Suppose $s_n$ is the number of different ways we can tile $5n$ inches of the strip. We need to find $s_{24}$. We saw $s_1 = 1, s_2 = 2$. How can we get to tile $5n$ inches, $n \geq 3$? We can either cover $5(n - 1)$ inches and lay one vertical tile, or we can tile $5(n - 2)$ inches and add two horizontal tiles. It follows that $s_n = s_{n-1} + s_{n-2}$. So here is how the values of $s_n$ evolve:
  - $1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025$,
- There are 75,025 different ways to tile the strip.