Outline

1. Triangles and their Angles

2. Areas, Angles, and More
In the triangle pictured above,
- The measure of the angle at $B$ is $37^\circ$.
- The measure of the angle at $C$ is $20^\circ$
- What is the measure of the angle at $A$?
Angling for angles

The figure below shows equilateral triangle $AED$ inside square $ABCD$. The segment $AC$ is a diagonal of the square. What is the measure of $\angle EFC$?
Solution

Because the triangle $AED$ is equilateral, all of its angles measure $60^\circ$. Since the segment $CA$ bisects a right angle, $\angle DAC$ measures $45^\circ$.

The two base angles of triangle $AFD$ add up to $45^\circ + 60^\circ = 105^\circ$, that leaves $\angle AFD$ no choice but to measure $180^\circ - 105^\circ = 75^\circ$. A very basic property of angles is that if two lines cross each other, then opposite angles are equal. Thus $\angle EFC$, and $\angle AFD$, being opposite angles created by the crossing of lines $AC$ and $DE$, must be equal. Since $\angle AFD$ measures $75^\circ$, so does $\angle EFC$.
In \( \triangle ABC \) the point \( D \) is on \( AC \) is such that \( |AB| = |AD| \). If \( \angle ABC - \angle ACB = 30^\circ \), find \( \angle CBD \). Justify your answer.
Solution

It may be convenient to give some names to the angles. It is traditional to use Greek letters for this purpose. Notice that because $\triangle ABD$ is isosceles, $\angle ABD = \angle ADB$. In the picture below I renamed the angles by

$$\alpha = \angle ACB, \quad \beta = \angle CBD, \quad \gamma = \angle ABD = \angle ADB.$$

We can rephrase the problem by: Find $\beta$ given that $(\beta + \gamma) - \alpha = 30$. Using that the sum of two angles of a triangle equal the angle supplementary to the third angle, we see that $\alpha + \beta = \gamma$ or $\beta = \gamma - \alpha$. From the given equation, $\beta = 30 - (\gamma - \alpha)$. Thus

$$2\beta = (\gamma - \alpha) + 30 - (\gamma - \alpha) = 30,$$

so $\beta = 15^\circ$. 
In the star shaped figure, the angle at $A$ measures $25^\circ$ and $\angle AFG = \angle AGF$. Find $\angle B + \angle D$. (AMC 8)
Solution

- $\angle B + \angle D = 180^\circ$ - red angle;
- blue angle $= 180^\circ$ - red angle.
- Thus $\angle B + \angle D = \text{blue angle}$.
- Because $\triangle AFG$ is isosceles, blue angle $= \text{green angle}$.
- $180^\circ = 25^\circ + \text{blue angle} + \text{green angle} = 25^\circ + 2 \times \text{blue angle}$.
- $\angle B + \angle D = \text{blue angle} = \frac{1}{2}(180 - 25) = 77.5^\circ$. 
A Counting Intermezzo

The integers 234, 417, 645 share a curious property: All three digits are different and one of the three digits is the average of the other two. How many three-digit numbers have this property? That is, how many three digit numbers are composed of three distinct digits such that one digit is the average of the other two?
Solution

Let us first get all sets of three digits \( \{a, b, c\} \) such that \( c = (a + b)/2, a < b \). For \( c \) to be a digit, either \( a, b \) are both even (10 choices) or both odd (also 10 choices). We have 20 choices in all. Each one of these choices can be arranged in 6 different ways. For example, from \((3, 5, 4)\) we get the integers

\[
345, 354, 453, 435, 534, 543.
\]

This gives a total of \( 6 \times 20 = 120 \) integers. But some of these will have a leading 0! If \( a = 0 \), then \( b \) is even, \( c = b/2 \), so we have to disregard the integers \( 021, 012, 042, 024, 036, 063, 084, 048 \), eight integers in all. The final answer is that there are 112 such integers.
Who Wins?

- Lines $m, n$ are parallel.
- Which triangle has the larger area? $\triangle ABC$ or $\triangle ABD$?
How many different isosceles triangles have integer side lengths and perimeter 23? (AMC 8)
Solution

If the sides of the triangle are $a, a, b$ then $a, b$ must satisfy $2a > b$ and $2a + b = 23$. All choices of $a, b$ satisfying these conditions work. We have

$$23 = 2a + b < 4a, \quad \text{so} \quad a > 23/4 = 5.75.$$  

Being an integer $a \geq 6$. next, since $b \geq 1$,

$$23 = 2a + b \geq 2a + 1, \quad \text{so} \quad 2a \leq 22,$$

hence $a \leq 11$. So $a$ is one of $6, 7, 8, 9, 10, 11$ giving a total of 6 such triangles.

Now $2a > b, 2a + b = 23$ implies $2a \geq \lceil 23/2 \rceil$, so $a \geq 6$. We also must have $a \leq 11$. But there are no other restrictions on $a$ so that we have a total of 6 such triangles.
Another AMC 8 problem.

In triangle $ABC$ point $E$ is on $AB$ with $|AE| = 1$, $|EB| = 2$. Point $D$ is on $AC$ so that $DE \parallel BC$ and point $F$ is on $BC$ so that $EF \parallel AC$. What is the ratio of the area of $CDEF$ to the area of $\triangle ABC$?
△$EBF$ is similar to △$ABC$. Because $|EB|/|AB| = 2/3$, the constant of proportionality is 2/3 and $[EBF] = (2/3)^2[ABC]$. Similarly, △$AED$ ∼ △$ABC$; since $|AE|/|AB| = 1/3$, we get $[AED] = (1/3)^2[ABC]$. Then

$$[ABF] + [AED] = \left(\frac{4}{9} + \frac{1}{9}\right)[ABC]; \quad \text{that is,} \quad [ABF] + [AED] = \frac{5}{9}[ABC].$$

Now

$$[CDEF] = [ABC] - ([ABF] + [AED]) = \frac{4}{9}[ABC].$$

The answer is $4/9$. 