Combinatorics II

Math Circle at FAU

September 30, 2023

Outline

Even Numbers

How many even 3 digit numbers have no repeating digits?

Solution: An even number has to end in 0, 2, 4,6, or 8. If the last digit is 2, 4, 6, or 8, the first digit can be anything but 0 or the last digit, so 8 choices. The second digit can be anything but the first of last digit, again 8 choices. So the number of even 3 digit numbers with no repeating digits ending in one of the four choices 2,4,6, or 8 is $4 \times 8 \times 8 = 256$.

If the last digit is 0 we have more choices; we have 9 choices for the first digit, 8 for the second, thus $9 \times 8 = 72$ choices. The answer is $256 + 72 = \boxed{328}$ such numbers.

Strings

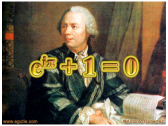
A *bit* is a binary digit, simply put, a 0 or a 1. A *n-bit string* is a string of *n* zeros and ones. Here are some examples:

- 111, 010, 100, 001 are 3-bit strings.
- 1111000, 0101010 are 6-bit strings.

Here is the question: How many 10 bit string have a sequence of EXACTLY 5 consecutive zeros?
For example, 0000011000, 1100000111, 0101000001 are such strings, 1000010000, 1000000111 are not.
Solution: Such a string could be at the very beginning, so the string must start with $000001\cdots$. That leaves 4 spaces open for 0's or 1', giving $2 \times 3 \times 2 \times 2 = 1$ strings. The sam argument shows there will be 16 strings ending in five zeros. Any other string can be identified by where the 1 preceding the string goes, which could be in one of positions 1, 2, 3, or 4. Such a string leaves only 3 free positions so we get an additional number of $4 \times (2 \times \times 2 \times 2) = 32$ strings. The answer is $16 + 16 + 32 =$ $\boxed{\text{64 strings.}}$

Meet one of the Greats

In how many different ways can one arrange the letters of EULER?



Solution:: $\frac{5!}{2!} = 60$ ways. See the analysis for Mississippi below.

Rolling Down the River

In how many different ways can one arrange the letters of MISSISSIPPI??

One way of looking at it

- Do you love solving equations?
- We can call x the number of ways the letters of Mississippi can be arranged. Once done this we can give new names to the repeated letters. Or use subindices. So we have an $S_1, S_2, S_3, S_4,\ I_1, I_2, I_3, I_4,\ P_1, P_2$. For each x ways we have of arranging letters leaving space for the S's, I's, P's, we have 4! ways of arranging $S_1 - S_4$, 4! ways to arrange $I_1 - I_4$, and 2! ways to arrange P_1, P_2. But once we have done this we have found all 11! ways of arranging the letters in
$$MI_1S_2S_2I_2S_3S_4I_3P_1P_2I_4$$
giving us the equation $x \times 4! \times 4! \times 2! = 11!$.
- The number of different ways is
$$x = \frac{11!}{4!4!2!} = 34,650 \quad \text{ways.}$$

Greeting the Queen

The Queen of Hearts will visit Snortle High, a school with only 30 students. Five of the students will be selected to present an award to the queen. In how many ways can the students be selected?

- $$\frac{P(30,5)}{5!} = \frac{30 \cdot 29 \cdot 28 \cdot 27 \cdot 26}{120} = 142,506 \quad \text{ways.}$$

Our Friends the Combinatorial Numbers

- The number of ways we can choose k objects from n (n *choose* k) is denoted by $\begin{pmatrix} n \\ k \end{pmatrix}$.
- Which of the following formulas do you think is right for $\begin{pmatrix} n \\ k \end{pmatrix}$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!} \tag{1}$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n^k}{k^n} \tag{2}$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{3}$$

$$\begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!} \tag{4}$$

Our Friends the Combinatorial Numbers

- The number of ways we can choose k objects from n (n *choose* k) is denoted by $\binom{n}{k}$.
- Which of the following formulas do you think are right for $\binom{n}{k}$?

$$\binom{n}{k} = \frac{n!}{k!} \tag{5}$$

$$\binom{n}{k} = \frac{n^k}{k^n} \tag{6}$$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \tag{7}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{8}$$

A Property of the Combinatorial Numbers

- Somewhere in the universe there is a school with a large number of students; very large; I won't tell you how large.
- The school needs to send some students to an event, and an administrator figured out that sending 30 students to the event can be done in

$$47,129,212,243,960$$

different ways. Then, only 29 students were to be sent. The administrator figured out that this could be done in

$$67,327,446,062,800$$

Then the principal asked the administrator did you take into account the new student; we now have one student more than before. The principal wished to know in how many ways one could select 30 students if the new student is also considered. The administrator's computer broke down. Can you help the administrator?

The Answer

$$\begin{array}{rr} \binom{n}{29} & 67,327,446,062,800 \\ \binom{n}{30} \quad + & 47,129,212,243,960 \\ \binom{n+1}{30} & 114,456,658,306,760 \end{array}$$

To select the 30 students from the pool enlarged by one student, one can just ignore the newcomer and select 30 students form the old group. That can be done in 47, 129, 212, 243, 960 different ways. The other way of selecting 30 students is to select 29 students from the old group, then add the new student. This can be done in 67,327,446,062,800 different ways. The total number of different ways is the sum of the two figures.

Can you Fill?

- Whatever n may be, $\binom{n}{n} = 1$
- Whatever n may be, $\binom{n}{0} = 1$
- Whatever n may be, $\binom{n}{1} = n$
- $$\begin{aligned} \binom{1}{1} &= 1 \\ \binom{2}{1} &= 2 \\ \binom{3}{1} &= 3 \\ \binom{3}{2} &= 3 \\ \binom{4}{2} &= 6 \\ \binom{7}{3} &= 35 \end{aligned}$$

A basic Property of Combinatorial Numbers

$$\boxed{\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.}$$

Pascal's Triangle

The previous property is behind Pascal's Triangle.

$$\begin{array}{ccccccccccc}
&&&&&1&&&&& \\
&&&&1&&1&&&& \\
&&&1&&2&&1&&& \\
&&1&&3&&3&&1&& \\
&1&&4&&6&&4&&1& \\
1&&5&&10&&10&&5&&1
\end{array}
\quad\Bigg|\quad
\begin{array}{ccccccccccc}
&&&&&\binom{0}{0}&&&&& \\
&&&&\binom{1}{0}&&\binom{1}{1}&&&& \\
&&&\binom{2}{0}&&\binom{2}{1}&&\binom{2}{2}&&& \\
&&\binom{3}{0}&&\binom{3}{1}&&\binom{3}{2}&&\binom{3}{3}&& \\
&\binom{4}{0}&&\binom{4}{1}&&\binom{4}{2}&&\binom{4}{3}&&\binom{4}{4}& \\
\binom{5}{0}&&\binom{5}{1}&&\binom{5}{2}&&\binom{5}{3}&&\binom{5}{4}&&\binom{5}{5}
\end{array}$$

- Can you fill in the next three rows?

$$\begin{array}{ccccccccccccccccc}
&&1&&6&&15&&20&&15&&6&&1&& \\
&1&&7&&21&&35&&35&&21&&7&&1& \\
1&&8&&28&&56&&70&&56&&28&&8&&1
\end{array}$$

- From filling these rows; in how many ways can we chose 3 elements from a set of 8?
  We start counting row entries beginning with 0. So the answer is the 3 entry from row 8, namely 35.

Subsets

- How many subsets does a set of 8 elements have, counting also the whole set and the empty set?
- Say the set is $\{a, b, c, d, e, f, g, h\}$.
- Idea: Assign to each subset an 8 digit binary string.

Solution:we can assign to each subset an 8-bit string. placing a 1 if the element (in the order given) is in the set, a 0 if not. Thus the full set (always considered a subset) gets the string 11111111. The subset $\{a, c, e\}$ the string 10101000, the empty subset, the subset without any elements gets the string 00000000. It is then easy to see, maybe, that the number 12

$$2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256.$$

And Now For Something Completely Different

- Compute the following sums
  1. $1+2+3+\cdots+99+100$.
     Solution:5050, using Gauss' method, as explained in the session. The general formula is
     $$1+2+\cdots+n=\frac{n(n+1)}{2}.$$
  2. $1+5+9+13+\cdots+401+405$.
     The number of terms to be added is 102. Let us write them as two rows and add vertically
     $$\begin{array}{cccccccc} & 1 & 5 & 9 & \cdots & 397 & 401 & 405 \\ & 405 & 401 & 397 & \cdots & 9 & 5 & 1 \\ \hline + & 406 & 406 & 406 & \cdots & 406 & 406 & 406 \end{array}$$
     That's a total of 102 times 406, but 102×406 counts every number twice. The answer is
     $$\frac{102\times 406}{2}=20,706$$

Back to Counting

Here is a problem from a competition I'll keep unmentioned. I changed it a bit. It's sort of sneaky.

How many subsets of the set $\{1, 2, 3, \ldots, 24, 25\}$ have the property that the sum of their elements is greater than 162.

Left as a challenge.