Combinatorics II

Math Circle at FAU

September 30, 2023

Outline

1. Back to Counting
2. Wending Our Way to Combinations
3. Freedom of Choice

Even Numbers

How many even 3 digit numbers have no repeating digits?

Strings

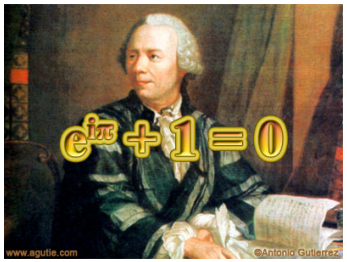
A *bit* is a binary digit, simply put, a 0 or a 1. A *n-bit string* is a string of n zeros and ones. Here are some examples:

- 111, 010, 100, 001 are 3-bit strings.
- 1111000, 0101010 are 6-bit strings.

Here is the question: How many 10 bit string have a sequence of EXACTLY 5 consecutive zeros?
For example, 0000011000, 1100000111, 0101000001 are such strings, 1000010000, 1000000111 are not.

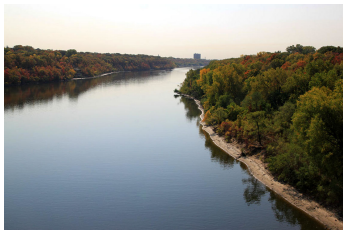
Meet one of the Greats

In how many different ways can one arrange the letters of EULER?



Rolling Down the River

In how many different ways can one arrange the letters of MISSISSIPPI??



Greeting the Queen

The Queen of Hearts will visit Snortle High, a school with only 30 students. Five of the students will be selected to present an award to the queen. In how many ways can the students be selected?

Our Friends the Combinatorial Numbers

- The number of ways we can choose k objects from n (n *choose* k) is denoted by $\left(\begin{array}{c} n \\ k \end{array} \right)$.
- Which of the following formulas do you think is right for $\left(\begin{array}{c} n \\ k \end{array} \right)$

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!} \quad (1)$$

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n^k}{k^n} \quad (2)$$

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n(n-1)\cdots(n-k+1)}{k!} \quad (3)$$

$$\left(\begin{array}{c} n \\ k \end{array} \right) = \frac{n!}{k!(n-k)!} \quad (4)$$

A Property of the Combinatorial Numbers

- Somewhere in the universe there is a school with a large number of students; very large; I won't tell you how large.
- The school needs to send some students to an event, and an administrator figured out that sending 30 students to the event can be done in

$$47,129,212,243,960$$

different ways. Then, only 29 students were to be sent. The administrator figured out that this could be done in

$$67,327,446,062,800$$

Then the principal asked the administrator did you take into account the new student; we now have one student more than before. The principal wished to know in how many ways one could select 30 students if the new student is also considered. The administrator's computer broke down. Can you help the administrator?

Can you Fill?

- Whatever n may be, $\binom{n}{n} =$ _____?
- Whatever n may be, $\binom{n}{0} =$ _____?
- Whatever n may be, $\binom{n}{1} =$ _____?
-

$$\binom{1}{1} = \text{\_\_\_\_\_}$$
$$\binom{2}{1} = \text{\_\_\_\_\_}$$
$$\binom{3}{1} = \text{\_\_\_\_\_}$$
$$\binom{3}{2} = \text{\_\_\_\_\_}$$
$$\binom{4}{2} = \text{\_\_\_\_\_}$$
$$\binom{7}{3} = \text{\_\_\_\_\_}$$

A basic Property of Combinatorial Numbers

$$\boxed{\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}.}$$

Pascal's Triangle

The previous property is behind Pascal's Triangle.

							1						$\binom{0}{0}$								
						1		1				$\binom{1}{0}$		$\binom{1}{1}$							
					1		2		1		$\binom{2}{0}$		$\binom{2}{1}$		$\binom{2}{2}$						
				1		3		3		1		$\binom{3}{0}$		$\binom{3}{1}$		$\binom{3}{2}$		$\binom{3}{3}$			
			1		4		6		4		1		$\binom{4}{0}$		$\binom{4}{1}$		$\binom{4}{2}$		$\binom{4}{3}$		$\binom{4}{4}$	
		1		5		10		10		5		1	$\binom{5}{0}$		$\binom{5}{1}$		$\binom{5}{2}$		$\binom{5}{3}$		$\binom{5}{4}$		$\binom{5}{5}$

- Can you fill in the next three rows?
- From filling these rows; in how many ways can we chose 3 elements from a set of 8?

Subsets

- How many subsets does a set of 8 elements have, counting also the whole set and the empty set?
- Say the set is $\{a, b, c, d, e, f, g, h\}$.
- Idea: Assign to each subset an 8 digit binary string.

And Now For Something Completely Different

- Compute the following sums
  1. $1 + 2 + 3 + \cdots + 99 + 100$.
  2. $1 + 5 + 9 + 13 + \cdots + 401 + 405$.

Back to Counting

Here is a problem from a competition I'll keep unmentioned. I changed it a bit. It's sort of sneaky.

How many subsets of the set $\{1, 2, 3, \ldots, 24, 25\}$ have the property that the sum of their elements is greater than 162.