Math Circle at FAU

Combinatorics-The Art of Counting

September 23, 2023

Outline

St. Ives

As I was going to St. Ives,
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits:
Kits, cats, sacks, and wives,
How many were there going to St. Ives?

Answer: 1 (Only I was going to St. Ives)

Let's be more precise

As I was going to St. Ives,
I met a man with seven wives,
Each wife had seven sacks,
Each sack had seven cats,
Each cat had seven kits:
Kits, cats, sacks, and wives,
How many did I meet on my way to St. Ives?

Answer: $7 \times 7 \times 7 \times 7 + 1 = 2402$.

Fundamental Principle of Counting

One or maybe more than one of the following is sometimes called the fundamental Principle of Counting.

- If the thing we are counting is the outcome of a multistage process then the number of possible outcomes is the sum of the number of choices for each stage.
- If the thing we are counting is the outcome of a multistage process then the number of possible outcomes is the product of the number of choices for each stage.
- If the thing we are counting is the outcome of a multistage process then the number of possible outcomes is the number of stages.
- The number of different ways one can perform m tasks, where each task has a number of choices in which it can be performed, is the product of all the possible choices for each task.

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Dressing Mario

Mario has

- three pairs of shoes, one black, one brown, one green.
- a pair of gray corduroy trousers, a pair of purple blue jeans, a pair of formal black trousers, and the must have red pants
- a yellow and a blue shirt

How many different outfits can Mario put together?

Answer: $3 \times 4 \times 2 = 24$.

Mario and Romeo

Let's make it a bit harder.

Mario and his twin brother Romeo share their clothes. They have

- three pairs of shoes, one black, one brown, one green.
- a pair of gray corduroy trousers, a pair of purple blue jeans, a pair of formal black trousers, and the must have red pants
- yellow and a blue shirt

In how many different ways can they dress?

Answer: $(3 \times 2) \times (4) \times (2 \times 1) = 144$.

Junior Journalists

A school has 100 students. The students want to publish a student newspaper. They have to choose an editor, a typesetter, and a reporter.

Assuming all students are equally qualified, in how many ways can this be done?

Answer: $100 \times 99 \times 98 = 970,200$ ways.

Student Government

Here is a similar one.

A school with 50 students wants to set up a student government. A president, vice president, secretary and treasurer have to be chosen.

In how many ways can this be done?

Answer: $50 \times 49 \times 88 \times 47 = 5,527,200$ ways.

Permutations

Permutations are fundamental mathematical objects. One usually writes

$$P(n,k)$$

for the different ways one can choose k objects from n.
Can you fill in the following the blanks?

$$\begin{aligned}
P(5,2) &= 5\times 4 = 20\\
P(100,3) &= 100\times 99\times 988 = 970,200\\
P(50,4) &= 50\times 49\times 48\times 47 = 5,527,200\\
P(5,5) &= 5\times 4\times 3\times 2\times 1 = 120\\
P(4,5) &= 0
\end{aligned}$$

The Factorial

- In general $P(n,k) = n(n-1)\cdots(n-k+1)$.
- An important case is if $k = n$.
- $P(n,n) = n(n-1)\cdots 2 \cdots 1 = n!$ (n factorial)

Some Factorials

Can you fill in the following blanks?

$$
\begin{aligned}
1! &= 1 \\
2! &= 2 \\
3! &= 6 \\
4! &= 24 \\
5! &= 120 \\
6! &= 720 \\
7! &= 5040
\end{aligned}
$$

The Good Luck Question

In how many different ways can you arrange the letters of the word FORTUNE ?

Answer: $7! = 5040$.

Professor Chang and His Books

Professor Chang has nine different language books lined up on a bookshelf: two Arabic, three German, and four Spanish. How many ways are there to arrange the nine books on the shelf keeping the Arabic books together and keeping the Spanish books together?

(A) 1440 **(B)** 2880 **(C)** 5760 **(D)** 1824400 **(E)** 362880

(AMC 8 2018)
vs
Answer: Treat the 2 Arabic books as being one book, and the 4 Spanish books as one book; so we look at the problem as if there were only 5 books. There are thus $5! = 120$ ways to arrange them. But now we look at th Arabic and Spanish books. There are $2! = 2$ ways of arraging them, $4! = 24$ ways of arranging the Spanish book. The total number of different ways the books can be arranged is therefore $120 \times 2 \times 24 = 5760$ ways. The correct answer is C.

Dancing the Night Away

At a party there are 10 boys and 10 girls. How many different ways can dance pairs be formed, assuming everybody dances and a dance pair has to consist of a boy and a girl (boys cannot dance with boys, girls cannot dance with girls).

Answer: Think of the boys as ordered. The first boy has a choice of 10 girls; the second of 9, and so forth. The number is
$10! - 3,628,800!$ ways.
How does the number change if boys are allowed to dance with boys, girls with girls?
Answer: Here we have 20 people to be paired; gender is now irrelevant. A first dancer has 19 choices for partner. That leaves 18 dancers, of which the next one has a choice of 17 partners, and so forth. The answer is
$19 \times 17 \times 15 \times 13 \times 11 \times 9 \times 7 \times 5 \times 3 \times 1 = 654,729,075$ ways.

Two Problems to Think About

- The following are hard problems. We will discuss them next time, or the time after that.
- There are almost certainly answers on the internet, but avoid looking them up. That would be cheating. And what's the point?

Lucky Seven

Can there exist 10,000 numbers divisible by 7, each ten digits long, and each one can be obtained from any of the others by reordering their digits?

The Famous Regions Problem

- If we place 15 points on the circumference of a circle and then draw all possible line segments having these points as end-points, if no three of these segments intersect in a point, into how many regions does this divide the disc?
- What is the general formula; telling you into how many regions n points divide the disc? It may be better to get the general formula first.
  The pictures below show the number of regions obtained by placing 2, 3, and 4 points on the circle.