Part 3: Required Proofs for Probability and Statistics Qualifying Exam

In what follows X_i's are always i.i.d. real random variables (unless otherwise specified).

You are allowed to use some well known theorems (like Lebesgue Dominant Convergence Theorem or Chebyshev inequality), but you must state them and explain how and where do you use them.

1. Prove that

$$\text{if } X_n \to X_0 \text{ in probability, then } X_n \to X_0 \text{ in distribution.}$$

Offer a connterexample for the converse.

2. Prove that

$$\text{if } E|X_n - X_0| \to 0. \quad \text{then } X_n \to X_0 \text{ in probability.}$$

Offer a connterexample for the converse.

3. We define $d_{BL}(X_n, X_0) = Sup_{H \in BL}|EH(X_n) - EH(X_0)|$, where BL is a set of all real functions that are Lipshitz and bounded by 1. Prove that

$$\text{if } d_{BL}(X_n, X_0) \to 0, \quad \text{then } P(X_n \le t) \to P(X_0 \le t)$$

for every t for which function $F(t) = P(X_0 \le t)$ is continuous.

4. Prove that

$$\text{if } X_n \to X_0 \text{ in probability and } Y_n \to Y_0 \text{ in distribution,}$$

then

$$X_n + Y_n \to X_0 + Y_0 \text{ in distribution.}$$

5. Prove that if $EX_i^2 < \infty$, then

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to E(X_1) \text{ in probability.}$$

6. (Count as two) Prove that if $E(|X_i|)$ exists, then

$$n^{-1}\sum_{i=1}^{n} X_i \to EX_1 \text{ in probability.}$$

7. Prove that if $EX_i^4 < \infty$, then

$$n^{-1}\sum_{i=1}^{n} X_i \to EX_1 \text{ a.s.}$$

Hint: Work with: $P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} |n^{-1}\sum_{i=1}^{n} X_i - EX_1| > \varepsilon)$.

8. (Count as two) Prove that if $E|X_i|^3 < \infty$, then

$$n^{-1/2}\sum_{i=1}^{n}(X_i - EX_1) \to Z \text{ in distribution,}$$

where Z is a centered normal random variable with $E(Z^2) = Var(X_i) = \sigma^2$.

9. Prove: For any $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$

$$E|XY| \le (E|X|^p)^{1/p} (E|X|^q)^{1/q}.$$

10. Prove that if

$$X_n \to X_0 \text{ in probability and } |X_i| \le M < \infty,$$

then

$$E|X_n - X_0| \to 0.$$

11. (Count as two) Let $F_n(t) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \le t\}}$ and $F(t) = P(X_i \le t)$ be a continuous function. Then

$$\sup_t |F_n(t) - F(t)| \to 0 \text{ in probability.}$$

12. Let X and Y be independent Poisson random variables with their parameters equal λ. Prove that $Z = X + Y$ is also Poisson and find its parameter.

13. Let X and Y be independent normal random variables with $E(X) = \mu_1, E(Y) = \mu_2, Var(X) = \sigma_1^2, Var(Y) = \sigma_2^2$. Show that $Z = X + Y$ is also normal and find $E(Z)$ and $Var(Z)$.

14. Let X_n converge in distribution to X_0 and let $f : R \to R$ be a continuous function. Show that $f(X_n)$ converges in distribution to $f(X_0)$.

15. Using only the Axioms of probability and set theory, prove that
a)
$$A \subset B \Rightarrow P(A) \leq P(B).$$

b)
$$P(X + Y > \varepsilon) \leq P(X > \varepsilon/2) + P(Y > \varepsilon/2).$$

c) If A and B are independent events, then A^c and B^c are independent as well.
d) If A and B are mutually exclusive and $P(A) + P(B) > 0$, show that
$$P(A|A \cup B) = \frac{P(A)}{P(A) + P(B)}.$$

16. Let A_i be a sequence of events. Show that
$$P(\cup_{i=1}^{\infty} A_i) \leq \sum_{i=1}^{\infty} P(A_i).$$

17. Let A_i be a sequence of events such that $A_i \subset A_{i+1}$, $i = 1, 2, ...$ Prove that
$$\lim_{n\to\infty} P(A_n) = P(\cup_{i=1}^{\infty} A_i).$$

18. Formal definition of weak convergence states that $X_n \to X_0$weakly if for every continuous and bounded function f :$R \to R$, $Ef(X_n) \to Ef(X_0)$. Show that:
$$X_n \to X_0 \text{ weakly } \Rightarrow P(X_n \leq t) \to P(X \leq t)$$
for every t for which the function $F(t) = P(X \leq t)$ is continuous.

19. (Borel-Cantelli lemma). Let A_i be a sequence of events such that $\sum_{i=1}^{\infty} P(A_i) < \infty$, then
$$P(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k) = 0.$$

20. Consider the linear regression model $Y = X\beta + e$, where Y is an $n \times 1$ vector of the observations, X is the $n \times p$ design matrix of the levels of the regression variables, β is an $p \times 1$ vector of the regression coefficients, and e is an $n \times 1$ vector of random errors. Prove that the least squares estimator for β is $\hat{\beta} = (X'X)^{-1}X'Y$.

21. Prove that if X follows a F distribution $F(n_1, n_2)$, then X^{-1} follows $F(n_2, n_1)$.

22. Let $X_1, \cdots, X_n$ be a random sample of size n from a normal distribution $N(\mu, \sigma^2)$. We would like to test the hypothesis $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. When σ is known, show that the power function of the test with type I error α under true population mean $\mu = \mu_1$ is $\Phi(-z_{\alpha/2} + \frac{|\mu_1 - \mu_0|\sqrt{n}}{\sigma})$, where $\Phi(.)$ is the cumulative distribution function of a standard normal distribution and $\Phi(z_{\alpha/2}) = 1 - \alpha/2$.

23. Let $X_1, \cdots, X_n$ be a random sample of size n from a normal distribution $N(\mu, \sigma^2)$. Prove that (a) the sample mean $\bar{X}$ and the sample variance S^2 are independent; (b) $\frac{(n-1)S^2}{\sigma^2}$ follows a Chi-squared distribution $\chi^2(n-1)$.