C_4-face-magic labelings on projective grid graphs

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For a graph $G = (V, E)$ embedded in the projective plane, let $\mathcal{F}(G)$ denote the set of faces of G. Then, G is called a C_n-face-magic projective graph if there exists a bijection $f : V(G) \to \{1, 2, \ldots, |V(G)|\}$ such that for any $F \in \mathcal{F}(G)$ with $F \cong C_n$, the sum of all the vertex labelings along C_n is a constant S. Let $x_v = f(v)$ for all $v \in V(G)$. We call $\{x_v : v \in V(G)\}$ a C_n-face-magic projective labeling on G. We consider the $m \times n$ grid graph, denoted by $\mathcal{P}_{m,n}$, embedded in the projective plane in the natural way. We show that for $m, n \geq 2$, $\mathcal{P}_{m,n}$ admits a C_4-face-magic projective labeling if and only if m and n have the same parity. Suppose $m \geq 3$ and $n \geq 3$ are odd integers. If m and n are distinct, then there are at least $2^{m/2+n/2-2}(\frac{m-1}{2})!(\frac{n-1}{2})!$ distinct C_4-face-magic projective labelings (up to symmetries on the projective plane) on $\mathcal{P}_{m,n}$. Also, $\mathcal{P}_{m,m}$ has at least $2^{m-3}((\frac{m-1}{2})!)^2$ distinct C_4-face-magic projective labelings (up to symmetries on the projective plane). Furthermore, $\mathcal{P}_{4,4}$ has exactly 144 distinct C_4-face-magic labelings (up to symmetries on the projective plane).
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