On Fractional Realizations of Tournament Score Sequences

David Brown, Bryce Frederickson, Braden Mindrum, Kaitlin Murphy*, Brent Thomas, Utah State University

Consider a tournament on $V = \{1, 2, \ldots, n\}$ with score sequence $\vec{s} = (s_1, s_2, \ldots, s_n)$, where $d^+(i) = s_i$. Define a *fractional realization of* $\vec{s}$ to be a complete loopless directed graph on V with each arc (i, j) weighted $\alpha_{ij} \in [0, 1]$ such that $\sum\limits_{j:i \neq j} \alpha_{ij} = s_i$ for all $1 \leq i \leq n$. Given a score sequence $\vec{s}$, we consider the polytope consisting of vectors of lexicographically indexed arc weightings, $\vec{\alpha} = (\alpha_{1,2}, \alpha_{1,3}, ..., \alpha_{n-1,n})$, whose corresponding directed graph is a fractional realization of $\vec{s}$. We show that the vertices of such a polytope have only integer entries. We also consider an interpretation of fractional directed graphs by viewing the arc weightings as probabilities. Given a fractional realization $D = (V, A)$ of a tournament score sequence $\vec{s}$, we define the tournament $T = (V, A')$ in which arc $(i, j) \in A'$ if and only if arc $(i, j) \in A$ is weighted greater than $\frac{1}{2}$. We say that T is a *probabilization* of $\vec{s}$ and that the score sequence of T is an *effective score sequence* of $\vec{s}$. Given a score sequence $\vec{s}$, we form the polytope consisting of all vectors $\vec{x}$ which are effective score sequences of $\vec{s}$ and show that the vertices of such a polytope are located at $\frac{1}{2}(\vec{s} + \vec{t})$ for each permutation $\vec{t}$ of $(0, 1, 2, ..., n-1)$. Furthermore, $\vec{x}$ is an element of the aforementioned polytope if and only if $2\vec{x} - \vec{s}$ satisfies Landau's conditions for a vector to be a score sequence of a tournament.

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