Total Non-negativity of Generating Functions

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A generating function is totally non-negative if each term in its expansion has a non-negative coefficient. We explore conditions on the coefficients of a generating function $f(x)$ that ensure that $f^{-1}(x)$, the compositional inverse of $f(x)$, is totally non-negative. For instance, if $f(x) = \sum_{r \in R} (-1)^{r-1} x^r / r!$, where R is a set of positive integers, a sufficient condition for $f^{-1}(x)$ to be totally non-negative is that $1 \in R$ and that the maximal consecutive intervals of R have even endpoints. For example, taking $R = \{1, 2\} \cup \{4\} \cup \{6, 7, 8, \ldots\}$ ensures that $f^{-1}(x)$ is totally non-negative. The proofs of total non-negativity are combinatorial, in the sense that we show that the coefficients are non-negative because each one is the size of a class of combinatorial objects.