Shameful Inequalities for List and DP Coloring of Graphs

Gunjan Sharma*, Hemanshu Kaul, Jeffrey A. Mudrock, Lake Forest College

The chromatic polynomial $P(G, k)$, introduced by Birkhoff in 1912, counts the number of proper k-colorings of a graph G. Enumerative analogues have since been developed for two generalizations of graph coloring: list coloring, via the list color function P_ℓ, and DP-coloring, via the DP color function P_{DP} and the dual DP color function P^*_{DP}. For any graph G and $k \in \mathbb{N}$, these functions satisfy $P_{DP}(G, k) \leq P_\ell(G, k) \leq P(G, k) \leq P^*_{DP}(G, k)$. In 2000, Dong settled the so-called Shameful Conjecture of Bartels and Welsh by proving that for any n-vertex graph G, $\frac{P(G,k+1)}{(k+1)^n} \geq \frac{P(G,k)}{k^n}$ for all $k \geq n-1$, while Seymour showed that the inequality can fail for smaller k. We study analogous inequalities for list and DP color functions. We prove that for any n-vertex graph G, $\frac{P_\ell(G,k+1)}{(k+1)^n} \geq \frac{P_\ell(G,k)}{k^n}$ and $\frac{P_{DP}(G,k+1)}{(k+1)^n} \geq \frac{P_{DP}(G,k)}{k^n}$ for all $k \in \mathbb{N}$. For the dual DP color function, we show that the inequality fails in general but holds for all $k \geq n - 1$ when G is an n-vertex complete bipartite graph.

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