On pet n-dimensional subtraction games

Mariya Naumova, Rutgers University

We consider impartial games in both normal and misère versions. Let $\mathcal{G}(x)$ and $\mathcal{G}^-(x)$ denote the normal and misère Sprague-Grundy values in a position x. Then, x is called an i-position if $\mathcal{G}(x) = i$ and an (i, j)-position if $\mathcal{G}(x) = i$ and $\mathcal{G}^-(x) = j$. A game is called: *forced* if it contains only $(0, 1)$- , $(1, 0)$-positions; *tame* if it contains only $(0, 1)$- , $(1, 0)$-, and (k, k)-positions; *pet* if it is tame with $k > 1$, that is, there are no $(0, 0)$- and $(1, 1)$-positions; *domestic* if it contains no $(0, k)$- and $(k, 0)$-positions with $k > 1$. Obviously, the families of forced, pet, tame, and domestic games are nested. We study n-dimensional subtraction games whose positions $x \in \mathbb{Z}_+^n$ are nonnegative n-vectors and possible moves are defined by a fixed set S of n-dimensional integer vectors $s = (s_1, \ldots, s_n)$ such that $\sum_{i=1}^n s_i > 0$. Game $\Gamma(S)$ is called *finite* if S is finite and *nonnegative* if each vector $s \in S$ is nonnegative. Two players move alternately and in one move $x \to y$, (from a position x to y) a player chooses any $s \in S$ such that $y = x - s$ is still nonnegative. The player who has to move but cannot loses the normal version of the game and wins its misère version. We prove that game $\Gamma(S)$ is pet if it is nonnegative and S has a (unique) minimum. Furthermore, we provide examples showing that both conditions are essential already for finite games. (For $n = 1$, the second condition holds automatically and the result is known.) We also prove that a nonnegative game $\Gamma(S)$ is pet if $|S| \leq 2$ and give examples of not domestic finite nonnegative subtraction games with $n = 2$ and $|S| = 3$.