130+ years of the Hadamard conjecture

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In 1893, Jacques Salomon Hadamard showed that if H is a square matrix of order n, with real entries of absolute value ≤ 1, then it satisfies the determinant bound $|\det(H)| \leq n^{\frac{n}{2}}$. Matrices that satisfy the equality in this determinant bound became known as Hadamard matrices. Alternatively, Hadamard matrices are $n \times n$ matrices with elements exclusively taken from $\{-1, +1\}$, s.t. $H \cdot H^t = nI_n$, where the superscript t denotes matrix transposition and I_n denotes the $n \times n$ identity matrix. Beyond the trivial Hadamard matrices of orders $n = 1, 2$, there is a necessary existence condition on the order, namely that $n \equiv 0 \pmod 4$. The sufficiency of this existence condition is the celebrated Hadamard conjecture, namely that there exists a Hadamard matrix of order n, for every n which is a multiple of four. Despite the fact that a plethora of constructions for Hadamard matrices are available, their collective distilled power does not suffice to provide a positive resolution of the Hadamard conjecture. We shall survey a number of important results on the Hadamard conjecture since its inception. We will also describe a "structured" form of the Hadamard conjecture, based on the concept of Legendre pairs, introduced in 2001. The conjecture that Legendre pairs exist for every admissible (odd) order, implies the Hadamard conjecture. We shall survey a number of important results on Legendre pairs in the past 20+ years.

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