A closure lemma for tough graphs and Hamiltonian degree conditions

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The closure of a graph G is the graph G^* obtained from G by repeatedly adding edges between pairs of non-adjacent vertices whose degree sum is at least n, where n is the number of vertices of G. The well-known Closure Lemma proved by Bondy and Chvátal states that a graph G is Hamiltonian if and only if its closure G^* is. This lemma can be used to prove several classical results in Hamiltonian graph theory. We prove a version of the Closure Lemma for tough graphs. A graph G is t-tough if for any set S of vertices of G, the number of components of $G - S$ is at most $t|S|$. A Hamiltonian graph must necessarily be 1-tough. Conversely, Chvátal conjectured that there exists a constant t such that every t-tough graph is Hamiltonian. The *t-closure* of a graph G is the graph G^{t*} obtained from G by repeatedly adding edges between pairs of non-adjacent vertices whose degree sum is at least $n - t$. We prove that, for $t \geq 2$, a $\frac{3t-1}{2}$-tough graph G is Hamiltonian if and only if its t-closure G^{t*} is. Hoàng conjectured the following: Let G be a graph with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$; then G is Hamiltonian if G is t-tough and, $\forall i < \frac{n}{2}$, if $d_i \leq i$ then $d_{n-i+t} \geq n - i$. This conjecture is analogous to the well known theorem of Chvátal on Hamiltonian ideals. Hoàng proved the conjecture for $t \leq 3$. Using the closure lemma for tough graphs, we prove the conjecture for $t = 4$. This is joint work with Cléophée Robin.

Keywords: cycles, Hamiltonian graphs