A Minimum Counterexample Proof of the Seymour Second Neighborhood Conjecture via the Graph Level Order

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We provide a constructive proof of Seymour's Second Neighborhood Conjecture (SSNC). In a minimal counterexample (MCE) environment, the possibilities reduce to the empty set. We introduce a coordinate system, the Graph Level Order (GLOVER), that represents any connected oriented graph via Breadth-First Search (BFS) based on their minimum out-degree node δ. This coordinate system has two key differences from BFS. First, it has set-theoretic ancestral memory. Second, by negating the conjecture, it implies that at every distance from the root, cycles must by present to prevent a MCE. The BFS is a spanning forest. It calls these cycles redundant. As these neighborhoods expand, the linear decrease of vertices and quadratic increase of rooted neighborhood capacity must inevitably collide. This exhausts arc capacity by rooted neighborhood R_i, where $i \leq \frac{\delta}{3}$. This collision forces the emergence of Seymour vertices. Furthermore, the complexity is $O(|V| + |E|)$.
Example ($\delta = 3$): Root v_0 has $N^+(v_0) = \{v_{1,1}, v_{1,2}, v_{1,3}\}$. To satisfy DNSP, $|N^{++}(v_0)|$ must be < 3. This forces every node in R_1 to form a cycle, allowing for $|R_2| = 2$. Each node in R_2 has out-degree requirements of 3, causing $N^{++}(v_{1,1}) \geq 3$. This results in $v_{1,1}$ as a Seymour vertex.

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