The Extremality of 2-partite Turán Graphs with Respect to the Number of Colorings

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Let q be a positive integer. A *q-coloring* of a simple graph G is a labeling of the vertices of G in at most q labels, called *colors*, such that no two adjacent vertices receive the same color. Let $P_G(q)$ denote the total number of q-colorings of a graph G. We will discuss an old problem by Linial and Wilf, to find the graphs with n vertices and m edges which maximize $P_G(q)$. The problem has been completely solved for $q = 2$, but the answer is still unknown for $q \geq 3$ and general n and m.

Lazebnik conjectured that among all graphs with the same number of vertices and edges as the r-partite Turán graph on n vertices, $T_r(n)$, the graph $T_r(n)$ is the only graph which obtains the most number of q-colorings for all integers $n \geq r \geq 2$ and $q \geq r$. Several cases of the conjecture have been solved for specific ranges of q and r. Toward the end of the talk, we willdiscuss the case when $r = 2$ and $q \geq 5$ is odd. A paper was recently published for this particular case when n is sufficiently large: https://epubs.siam.org/doi/10.1137/22M1511990.