Reciprocals of Thinned Exponential Series

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The power series (about 0) of $e^{-x} = 1 - x + x^2/2! - x^3/3! + \cdots$ has the property that the reciprocal has a power series given by $1/e^{-x} = e^x = 1+x+x^2/2!+x^3/3!+\cdots$, and this power series is non-negative (i.e. all coefficients are non-negative). If we truncate the series for e^{-x} following an odd power $-x^{2k+1}/(2k+1)!$, then this non-negativity property still holds for the reciprocal of the resulting Taylor polynomial. If, however, we truncate the series following an even power $x^{2k}/(2k)!$, the resulting series of the reciprocal has some negative coefficients. Gessel gave a combinatorial explanation for the non-negativity phenomenon.

We'll extend this from truncates to classes of "thinned" exponential series (obtained from the power series of e^{-x} by deleting some set of terms). In particular, we consider $f_A(x) = 1+\sum_{n\in A}(-1)^n x^n/n!$ for certain subsets A of $\{1, 2, 3, \ldots\}$, and give a combinatorial proof why these subsets have a reciprocal series that is non-negative. The truncates use $A = \{1, \ldots, m\}$. One class of A we will consider are those A which, when written in terms of maximal intervals of consecutive integers, have the ends of each interval being odd: for example, $A = \{1, 2, 3\} \cup \{5, 6, 7\} \cup \{11\} \cup \{35, 36, 37, \ldots\}$. We'll include some open questions.