The 5-cube cut number, cut-complexes, and a vertex coloring game

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The hypercube cut-complexes are convex geometric presentations for threshold Boolean functions. The latter Boolean functions have been the core elements of study in threshold logic, and the original bases for modern deep learning. These complexes are in fact isometric subgraphs of the geometric d-cube, which are separable from the rest of the hypercube by a hyperplane in R^d. They are also very closely connected to the cut number of the d-cube. The cut number $S(d)$ is the minimum number of hyperplanes in R^d that slices all the edges of the d-cube. The problem was originally posed by P. O'Neil in 1971 and then appeared as one of Victor Klee's unresolved problems in his invited talk at the CCCG-1999 conference. Many other pioneers in convex or discrete geometry, including B. Grünbaum, M. Saks, and Z. Füredi have also raised the problem in different contexts. The identity $S(d) = d$ has been well-known for $d \leq 4$, since Emamy presented two different solutions for the 4-cube, in 1986 and 1988. On the other hand, the 5-cube problem appeared to be much harder. It was not until the year 2000 that Sohler and Ziegler obtained a computational proof for $S(5) = 5$. Moreover, finding a theoretical and computer-free proof for the problem remains a challenging open problem. Based on a recent paper by the authors, we present a vertex coloring game over the 5-cube that will be a fundamental basis for proving $S(5) = 5$.

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