Efficient computing of bounded solutions to linear Diophantine equation with Euler's totient function

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Let $\varphi(n)$ and $\tau(n)$ denote Euler's totient function and the number of divisors of n, respectively. We propose an efficient computational method for finding all solutions $n \le U$ to the Diophantine equation

$$a\varphi(n) = bn + c,$$

where integer coefficient a, b, c and an upper bound U are given. Our method is implemented in the SageMath computer algebra system within the framework of recursively enumerated sets and natively benefits from MapReduce parallelization.

Using our method, we discovered new solutions to many published equations and closed gaps between known large solutions. By incorporating additional constraints, including bounds for the number of prime divisors, we discovered a composite solution $n > 4$ to the congruence $\varphi(n)\tau(n)+2 \equiv 0 \pmod{n}$, thus answering a long-standing open question posed by Subbarao in 1974. Furthermore, we established that our solution

$$n = 25643985470955911102 = 2 \cdot 61 \cdot 67 \cdot 6803 \cdot 29027 \cdot 15887233$$

is the smallest such solution.

Keywords: Euler's totient function, Lehmer's totient problem, Subbarao's congruence, MapReduce