Compositions with Descents at Odd Numbered Plus Signs

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For a positive integer n, a composition of n is an ordered sum $x_1+x_2+\ldots+x_m$, where $m \leq n$, and each summand x_i, for $1 \leq i \leq m$, is a positive integer. There are 2^{n-1} compositions of n. Now, we are interested in the compositions of n, where, for $1 \leq i \leq \lfloor \frac{m}{2} \rfloor$, $x_{2i-1} > x_{2i}$. So the consecutive summands at an odd numbered plus sign (as a composition is read from left to right) form a descent. If $n = 9$, for example, then of the 256 compositions of 9, only 34 satisfy this condition — three of them being $6+3$, $5+2+2$, and $4+2+3$. In general, the number of such compositions of n is given by the nth Fibonacci number.
For a given value of n, we derive and solve recurrence relations to determine the following for these compositions of n: (1) The total number of summands; (2) The numbers of summands in even and odd positions; (3) The sums of all the first, second, third, and fourth summands; (4) The number that have three and four summands; and, (5) The number of times 1 appears as a summand.

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