

Analysis Qualifier Exam

Instructions: Write complete solutions with clear reasoning.

Name: _____ **ID:** _____

Problem 1. Recall that a point in a metric space Y is isolated if it is not a limit point of Y . Let Y be a metric space with an isolated point y .

- (a) Show that the only connected subset of Y containing y is $\{y\}$.
- (b) Prove that any continuous function $f : X \rightarrow Y$ where X is connected and $y \in f(X)$ is constant.

Problem 2. Show that compact subsets of metric spaces are closed.

Problem 3. Let $E \subset X$ where (X, d) is a metric space. If x is a limit point of E , then there exists a sequence $\{x_n\}$ in E that converges to x .

Problem 4. Let A be a non-empty subset of X . Define the distance from $x \in X$ to A by

$$d(x, A) = \inf_{y \in A} d(x, y).$$

Let E be a nonempty subset of X . Prove that $x \in \overline{E}$ if and only if $d(x, E) = 0$.

Problem 5. Let $f_n(x) = \frac{x}{x+n}$ for $x \geq 0$.

- (a) Show that $f_n(x) \rightarrow 0$ pointwise on $[0, \infty)$.
- (b) Show that the convergence is uniform on $[0, t]$ for each $t > 0$.
- (c) Show that the convergence is not uniform on $[0, \infty)$.

Problem 6. Let f and g be real-valued continuous functions on a metric space X (i.e., $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}$). Prove that $f + g$ is continuous on X .

Problem 7. Let f be Riemann integrable on $[a, b]$. For each $x \in [a, b]$, define the function $F(x)$ as:

$$F(x) = \int_a^x f(t) dt.$$

Show that $F(x)$ is uniformly continuous on $[a, b]$.

Problem 8. Prove the following statement:

Let $\{E_k\}$ be a countable collection of Lebesgue measurable sets. Then

$$E = \cup E_k$$

is Lebesgue measurable and $m(E) \leq \sum m(E_k)$.