Algebra Qualifying Exam

Florida Atlantic University, January 9, 2026, 9 am – 12 pm

Student Name: ____________________

1. Define the following terms: ring, left ideal, right ideal, two-sided ideal, prime ideal of a commutative ring with $1 \neq 0$, maximal ideal of a commutative ring with $1 \neq 0$.

   If R is a ring and $I \subseteq R$ a subring, what property of I is necessary and sufficient to turn R/I into a ring such that the projection $\pi\colon R \to R/I$ mapping r to $r + I$ be a ring homomorphism?

2. Let G be a finite simple group. Suppose G contains a subgroup H of prime index p. Prove that p is the largest prime divisor of $|G|$ and, moreover, p^2 does not divide $|G|$. (Hint: Let G act on the left cosets of H in G.)

3. Let m, n be two coprime positive integers and G a group of order mn with identity 1. Suppose that, for all $a, b \in G$, we have $a^m b^m = b^m a^m$ and $a^n b^n = b^n a^n$. Let M, N be the subgroups of G generated respectively by m-th powers and n-th powers of elements of G.

   (a) Show that M and N are normal subgroups of G.
   (b) Show that if $g \in M$ then $g^n = 1$ and if $g \in N$ then $g^m = 1$.
   (c) Deduce that $M \cap N = \{1\}$.
   (d) Show that $G = MN$. (Hint: Use Bézout's identity expressing $\gcd(m, n)$ as a linear combination of m and n with coefficients in $\mathbb{Z}$.)
   (e) Deduce that G is abelian.

4. Prove the following generalization of Eisenstein's criterion: Let $p \in \mathbb{N}$ be prime and $f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 \in \mathbb{Z}[x]$. Let $k \in \mathbb{N} \cup \{0\}$ such that $p \nmid a_n$, $p \mid a_{n-k-i}$ for $1 \leq i \leq n-k$ and $p^2 \nmid a_0$. If $f(x) = g(x)h(x)$ with $g, h \in \mathbb{Z}[x]$, show that $\min(\deg g, \deg h) \leq k$. Use it to show that, for $n > 1$, $x^n + 5x^{n-1} + 3$ is irreducible in $\mathbb{Z}[x]$. (Hint: Reduce modulo p.)

5. Let $\mathbb{F}_{64}$ be a field with 64 elements. Draw its lattice of subfields. List the irreducible polynomials of $\mathbb{F}_2[x]$ of degree 2 and 3. How many irreducible polynomials in $\mathbb{F}_2[x]$ of degree 6 are there? Show that $\mathbb{F}_{64}$ is isomorphic to $\mathbb{F}_2[x]/(x^6 + x + 1)$.

6. Let $p_1, p_2, p_3 \in \mathbb{N}$ be distinct primes. Let $K_i = \mathbb{Q}(\sqrt{p_i})$ for $i = 1, 2, 3$.

   (a) Let $d_1, d_2 \in \mathbb{Z} \setminus \{0\}$. Show that $\mathbb{Q}(\sqrt{d_1}) = \mathbb{Q}(\sqrt{d_2})$ if and only if $d_1 d_2$ is a perfect square.
   (b) Show that $[K_i : \mathbb{Q}] = 2$ for $i = 1, 2, 3$.
   (c) Let $K = K_1 K_2$. Show that $[K : \mathbb{Q}] = 4$.
   (d) Show that K is a Galois extension of $\mathbb{Q}$. What is the isomorphism type of its Galois group? Describe its elements.
   (e) How many subfields of K of degree 2 over $\mathbb{Q}$ are there? List them all as $\mathbb{Q}(\sqrt{m})$ where m is a squarefree integer.
   (f) Let $F = K_1 K_2 K_3$. Show that $[F : \mathbb{Q}] = 8$.
   (g) Show that F is Galois over $\mathbb{Q}$ and find the isomorphism type of its Galois group.