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Current Directions and Emerging Opportunities for Research in Algebra

White Paper on the Panel Discussion at the 2003 Weekend Algebra Meeting

The 2003 Weekend Algebra Meeting took place November 7–9 on the Boca Raton campus of Florida Atlantic University. Many of the participants were graduate students and young Ph.D.'s. The purpose of the panel discussion at the 2003 meeting has been to increase awareness for current directions and emerging opportunities in Algebra.

The three panel speakers, Dr. Bruce Olberding (New Mexico State U), Dr. Fred Richman (FAU), and Dr. Bill Wickless (U Connecticut), each offered a 20 minute presentation; chairman of the session, which took place on Saturday, November 8, 1–3 p.m., was Dr. Jim Brewer (FAU). The following is a subjective recollection by the editors (LK and MS) of some of the points made by the panel speakers.

Fred Richman: Uncharted Territory Ahead

Fred compared the challenge of charting the future of algebra to Hilbert's address at the Paris conference. He questioned the wisdom of making a guiding pronouncement about current directions in algebra. He recalled that, years ago when category theory was a hot topic, Saunders MacLane wondered whether it would end up like lattice theory, which was pretty much dead at the time. Yet now lattice theory again seems to be a hot topic.

So how can we possibly find out about which directions to go? Fred recommended to look out for open problems routinely. Also to pay attention to developments in mathematics which may impact Algebra. As for example set theory and large cardinals did when they permeated various branches of mathematics.

What is the real impact of computers? True, they stimulate interest in recursive functions and in numerical approximations. Beyond that, computability itself is of interest in algebra as computation-in-principle differs starkly from actual computability as the well known theorem that *every nonconstant polynomial is a product of primes* illustrates: The proof follows easily from the definitions, and yet, the result is hard to verify in large examples. Coming back to the question about which directions to go, one can have computability in mind while doing mathematics, or one might explore the impact of a different set of logic rules on algebra, like for example restricted versions of the law of excluded middle.

Bruce Olberding: Link up with Key Mathematics Areas

Bruce presented to us surprising examples for the interplay between algebra, logic, geometry, and number theory that came up in his research on non-noetherian commutative rings.

The ring $\mathrm{Int}(\mathbb{Z})$ of integer valued polynomials is a well-studied example of a non-noetherian commutative ring, but there are many such rings, even between $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$. Some are Bézout domains which are characterized by the property that finitely generated torsion-free modules are free.

Open Question: If R is a Bézout domain, is every finitely presented R-module a direct sum of cyclic R-modules?

It turns out that the statement is equivalent to a first order property in the domain R, and so the open question invites us to study the role of mathematical logic in this context.

According to a result by Nagata, the intersection of finitely many valuation domains is a Bézout domain. Consider the case that K is a field of finite transcendence degree n over $\mathbb{R}$. Then the intersection H of all valuation rings in K with residue field $\mathbb{R}$ is a Prüfer domain, the *real holomorphy ring* of K. According to a result by Schulting, Swan, and Kucharz, H contains an ideal that requires at least $n+1$ generators. We ask:

Open Questions: What is the ideal structure of holomorphy rings? Is it possible to give an "algebraic" proof of this result?

It follows that there are elements $f_1, \ldots f_{n+1}$ in K such that for each

$k > 0$, the sum $f_1^{2k} + \cdots + f_{n+1}^{2k}$ cannot be written as a sum of $2k$-th powers of fewer elements (Kucharz).

Open Question: Are there more connections between the ideal theory of the holomorphy ring H and "sums of squares" in K, orderings on K, and the geometry of (blowups of) real points on projective models of K?

Bill Wickless: Keep the Classics in Mind

In his presentation, Bill focused on abelian group theory. He indicated ideas which he finds of current interest by listing a few of his favorite research problems.

A group G is called A-*projective* if G is a direct summand of a direct sum of copies of A. One research problem would be to find conditions such that G would be A-projective for various groups A.

What can one say about abelian groups G and H such that the endomorphism rings $E(G) \cong E(H)$? At one extreme, if G and H are torsion, then $G \cong H$, while at the other extreme, there seems to be no necessary connection when G and H are torsion-free. What can one say if G and H are mixed groups?

Groups G and H are called multi-isomorphic if $G^n \cong H^n$ for all integers $n \geq 2$; logically, multi-isomorphism is somewhere between isomophism and quasi-isomorphism. Another research problem would be to explore conditions such that two groups be multi-isomorphic.