MATH DAY 2008 at FAU

Competition B-Teams

SOLUTIONS

NOTE:

1. Enter the name of your team on the answer sheet. **Only one answer sheet per team should be handed in.** Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Twenty of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer which you enter directly beside the problem number on the answer sheet. Make sure you write clearly.
3. In the multiple choice questions, the option NA stands for "None of the previous answers is correct."
4. In all questions, i stands for the imaginary unit; $i^2 = -1$.
5. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
6. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1* The remainder of the polynomial $x^4 - 3x^3 + ax^2 + bx + 3$ when divided by $x - 1$ is 5. When divided by $x + 1$ the remainder is also 5. **Determine** b. Enter your answer directly on the answer sheet.

SOLUTION. We recall that if $p(x)$ is a polynomial, then the remainder of dividing $p(x)$ by $x - a$ is $p(a)$. Evaluating the polynomial at 1 and at -1 and setting the results equal to 5 leads to the equations

$$\begin{cases} a + b = 4 \\ a - b = -2 \end{cases}$$

Solving we get $b = 3$.

The number to be entered should be **3.** ■

2. For $x^2 + 2x + 5$ to be a factor of $x^4 + px^2 + q$, the values of p and q must be, respectively:

(A) $-2, 5$ **(B)** $5, 25$ **(C)** $10, 20$ **(D)** $6, 25$ **(E)** $14, 25$

SOLUTION. By necessity, the other factor would have to be also a quadratic polynomial. If $(x^2+2x+5)(ax^2+bx+c) = x^4 + px^2 + q$, multiplying out and equating coefficients, one gets:

$$a = 1, \quad 2a + b = 0, \quad 5a + 2b + c = p, \quad 5b + 2c = 0, \quad 5c = q.$$

Solving in the order

$$a = 1, \quad 2a + b = 0, \quad 5b + 2c = 0, \quad 5a + 2b + c = p, \quad 5c = q.$$

one gets

$$a = 1, \quad b = -2, \quad c = 5, \quad p = 6, \quad q = 25.$$

The correct solution is **D.** ■

3. Let $1, x_1, x_2, \cdots, x_{10}$ be the roots of the equation $x^{11} = 1$, where $x_1, x_2, \cdots, x_{10}$ are the ten distinct complex non-real roots. Find $(1 - x_1^2)(1 - x_2^2)\cdots(1 - x_{10}^2)$.

(A) 0 **(B)** 1 **(C)** 10 **(D)** 11 **(E)** NA

SOLUTION. Because $1, x_1, \ldots, x_{10}$ are the roots of $x^{11} - 1 = 0$, one has

$$x^{11} - 1 = (x - 1)(x - x_1)\cdots(x - x_{10}).$$

But one also has

$$x^{11} - 1 = (x - 1)(x^{10} + x^9 + \cdots + x^2 + x + 1).$$

Thus

$$(x - x_1)\cdots(x - x_{10}) = x^{10} + x^9 + \cdots + x^2 + x + 1 \tag{1}$$

for all x. Now

$$(1 - x_1^2)(1 - x_2^2)\cdots(1 - x_{10}^2) = (1 - x_1)(1 - x_2)\cdots(1 - x_{10})(1 + x_1)(1 + x_2)\cdots(1 + x_{10})$$

and applying (1) with $x = 1$ we get

$$(1 - x_1)(1 - x_2)\cdots(1 - x_{10}) = \underbrace{1 + 1 + \cdots + 1}_{11 \text{ times}} = 11;$$

applying the same equation with $x = -1$

$$(1 + x_1)(1 + x_2)\cdots(1 + x_{10}) = (-1)^{10}(-1 - x_1)(-1 - x_2)\cdots(-1 - x_{10}) = (-1)^{10}(1 - 1 \pm \cdots - 1 + 1) = 1.$$

Thus

$$(1 - x_1^2)(1 - x_2^2)\cdots(1 - x_{10}^2) = 11 \cdot 1 = 11.$$

The correct solution is **D.** ■

4. The minimum value of $x^4 + 4x^3 + 6x^2 + 4x + 3$ is

(A) 3 **(B)** 2 **(C)** 1 **(D)** 0 **(E)** NA

SOLUTION.

$$x^4 + 4x^3 + 6x^2 + 4x + 3 = (x + 1)^4 + 2$$

from which it is clear that the minimum value achieved is 2.

The correct solution is **B.** ■

5. The sum of the reciprocals of the roots of the equation $x^2 + px + q = 0$, where $p, q \neq 0$, is:

(A) $-p/q$ **(B)** q/p **(C)** p/q **(D)** $-q/p$ **(E)** pq **(F)** NA

(The reciprocal of a number $x \neq 0$ is the number $1/x$)

SOLUTION. If x_1, x_2 are the roots of the equation $x^2 + px + q = 0$, then $x_1 + x_2 = -p$, $x_1 x_2 = q$ so that

$$\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1 + x_2}{x_1 x_2} = -\frac{p}{q}.$$

The correct solution is **A.** ■

6. How many distinct solutions does the equation $x^2 - 8[x] + 7 = 0$ have? Here $[x]$ represents the largest integer not exceeding the real number x, also called the *floor* of x. For example, $[\sqrt{2}] = 1$, $[\pi] = 3$, $[-1.2] = -2$, and $[4] = 4$.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

SOLUTION. The equation $x^2 - 8x + 7 = 0$ has the roots 1, 7 which, being integers, are also roots of $x^2 - 8[x] + 7 = 0$. Moreover, $x^2 - 8x + 7 > 0$ if $x > 7$ or $x < 1$ and since $x^2 - 8[x] + 7 \geq x^2 - 8x + 7$ (because $[x] \leq x$) we see that if there are any other roots of our equation they cannot be integers and they must satisfy $1 < x < 7$. We can now break up the interval $(1, 7)$ (ignoring the integers in the interval) into the intervals $(1, 2), (2, 3)$, etc. and go interval by interval. We get:

(a) $1 < x < 2$, the equation becomes $x^2 - 1 = 0$, which has no solution in the interval.
(b) $2 < x < 3$, the equation becomes $x^2 - 9 = 0$, which has no solution in the interval.
(c) $3 < x < 4$, the equation becomes $x^2 - 17 = 0$, which has no solution in the interval.
(d) $4 < x < 5$, the equation becomes $x^2 - 25 = 0$, which has no solution in the interval.
(e) $5 < x < 6$, the equation becomes $x^2 - 33 = 0$, which has one solution in the interval, namely $x = \sqrt{33}$.
(f) $6 < x < 7$, the equation becomes $x^2 - 41 = 0$, providing another solution since $6 < \sqrt{41} < 7$.

Therefore, there are exactly four solutions: $x = 1, 7, \sqrt{33}, \sqrt{41}$.

The correct solution is **D.** ■

7.* If the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers, determine the sum of the first $4n$ positive integers. Enter your answer directly on the answer sheet.

SOLUTION. The sum of the first m positive integers is $m(m + 1)/2$. We thus have that

$$\frac{3n(3n + 1)}{2} = \frac{n(n + 1)}{2} + 150.$$

Solving for n we get the equation $8n^2 + 2n - 300 = 0$; thus $n = 6$, $4n = 24$. The sum of the first 24 positive integers is $24 \cdot 25/2 = 300$.

The number to be entered should be **300.** ■

8. Three digit numbers are formed using only odd digits. The sum of all such three digit numbers is:

(A) 19375 **(B)** 34975 **(C)** 6253 **(D)** 34975 **(E)** 69375

SOLUTION. There are $5 \times 5 \times 5 = 125$ such numbers. We can think of all them listed in a column, "odometer" style (the last digit changes faster then the second digit, the second faster than the first):

$$\begin{array}{c} 111 \\ 113 \\ \vdots \\ 119 \\ 131 \\ \vdots \\ 199 \\ 311 \\ \vdots \\ 999 \end{array}$$

Each column by itself contains the numbers 1,3,5,7,9, each number repeated 25 times. Thus each column adds up to $25 \times (1 + 3 + 5 + 7 + 9) = 625$ and the total sum is $625 + 10 \times 625 + 100 \times 625 = 69375$.

The correct solution is **E.** ■

9. If you have an unlimited supply of 3-cent and 8-cent postage stamps, what is the largest value (in cents) that you can**not** place on an envelope?

(A) 10 **(B)** 13 **(C)** 17 **(D)** 19 **(E)** NA

SOLUTION. One sees quite fast that $17 = 3 \cdot 3 + 8$, $19 = 3 + 2 \cdot 8$ while there is no way to write 13 as a sum of (non-negative) multiples of 3 and 8. Thus the answer has to be either B or E, and it is a good bet that it is B.

For a more careful approach, we are asking what is the **largest** m such that the equation $3a + 8b = m$ can only be solved if at least one of a, b is negative. While there is a whole theory behind such equations, a bit of simple reasoning will do in this case, specially since we already know that $m \geq 13$. Let's say we want to put m cents in stamps onto an envelope, using only 3 and/or 8 cent stamps. If m is a multiple of 8 it can of course be done. Suppose it isn't. Then one of $m, m - 8$ or $m - 16$ is a multiple of 3. In fact, if m is not a multiple of 3, then m has a remainder of 1 or 2 when divided by 3. If the remainder is 1, then $m - 16$ is a multiple of 3; otherwise $m - 8$ is a multiple of 3. This tells us at once that if $m \geq 16$, then we can get m cents in postage using only 3 and 8 cent stamps. It leaves 15 and 14 as the only alternatives to our answer of 13. But 15 is a multiple of 3, $14 - 8 = 6$ is a multiple of 3. Thus 13 is the number we are looking for.

The correct solution is **B.** ■

10. There is a group of children, in which the oldest is 13 and one of them is 10. The sum of their ages is 50. Also, the ages of the children except the one who is 10 form an arithmetic progression. Find the number of children in the group.

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NA

SOLUTION. Suppose we have n children, **excluding** the ten year old, so that there are $n+1$ children in all. Since the ages of these n children are in geometric progression; since the oldest is 13, and since the sum of all ages is $50 - 10 = 40$, we get if r is the ratio of the progression:

$$13 + (13 - r) + \cdots + (13 - (n-1)r) = 40.$$

(It is $13 - (n-1)r$ and not $13 - nr$ because we start counting at 0.) Rearranging we get

$$n(26 - (n-1)r) = 80.$$

Thus n must be a divisor of 80. The youngest child has age $13 - (n-1)r$; we must have $(n-1)r \leq 13$ (for the child to be born), thus $26 - (n-1)r \geq 13$ and this limits n to the range $1 \leq n \leq 6$; otherwise $n(26 - (n-1)r) > 80$. But the factor of n in $n(26 - (n-1)r) = 80$ is ≤ 26, so $n \geq 4$. The only divisor of 80 in the range $4 \leq n \leq 6$ is $n = 4$. Then $r = 2$ and the children have ages (in descending order) $13, 11, 10, 9, 7$.

The correct solution is **A.** ■

11. How many zeros does 100! end in? For example, $10! = 3628800$ ends in two zeros.

(A) 24 **(B)** 26 **(C)** 28 **(D)** 30 **(E)** NA

(If n is a positive integer, then $n! = 1 \cdot 2 \cdots (n-1)n$ is the product of all integers from 1 to n)

SOLUTION. $100! = 1 \cdot 2 \cdot 3 \cdots 99 \cdot 100$. If we remove all the multiples of 5, the resulting product is not divisible by 10, and has no trailing zeros. There being an ample supply of even numbers, each multiple of 5 that is not a multiple of 25 contributes one zero. each multiple of 25 contributes 2 zeros. There are 20 multiples of 5 in all; of these four (25, 50, 75 and 100) are also multiples of 25. The total number of trailing zeros is thus $16 + 2 \cdot 4 = 24$.

The correct solution is **A.** ■

12. 873 digits are used to number the pages of a book consecutively from page 1. How many pages are there in the book?

(A) 255 **(B)** 290 **(C)** 320 **(D)** 327 **(E)** NA

SOLUTION. Let x be the total number of pages. For the first nine pages we need 9 digits. The ninety pages from page 10 up to and including page 99, require two digits per page, so at page 99 we would have used a total of 189 digits.

Clearly $x > 99$. Suppose we do not need to use four digit numbers, then the equation for x is $3(x-99) = 873 - 189$ or $x = 327$.

The correct solution is **D.** ■

13. When simplified the product

$$\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right)$$

becomes:

(A) $\frac{1}{n}$ **(B)** $\frac{2}{n}$ **(C)** $\frac{2(n-1)}{n}$ **(D)** $\frac{2}{n(n+1)}$ **(E)** $\frac{3}{n(n+1)}$

SOLUTION. We have

$$\begin{aligned}\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{5}\right)\cdots\left(1-\frac{1}{n}\right) &= \left(\frac{2}{3}\right)\left(\frac{3}{4}\right)\left(\frac{4}{5}\right)\cdots\left(\frac{n-1}{n}\right)\\ &= \left(\frac{2}{\not{3}}\right)\left(\frac{\not{3}}{\not{4}}\right)\left(\frac{\not{4}}{\not{5}}\right)\cdots\left(\frac{n \not{-} 1}{n}\right)\\ &= \frac{2}{n}.\end{aligned}$$

The correct solution is **B.** ■

14. If the area of a circle is doubled when its radius r is increased by n, then r equals:

(A) $(\sqrt{2}+1)n$ **(B)** $(\sqrt{2}-1)n$ **(C)** n **(D)** $(2-\sqrt{2})n$ **(E)** NA

SOLUTION. The equation relating r and n is $\pi(r+n)^2 = 2\pi r^2$ from which one gets $r^2 - 2rn - n^2 = 0$; solving for r and considering that $r > 0$, one gets $r = n(1+\sqrt{2})$.

The correct solution is **A.** ■

15. Let $a > 1$ and suppose that x is a **positive** solution of $6a^x + 6a^{-x} = 13$. Then x equals

(A) $\log_a 3$ **(B)** $\log_a 2$ **(C)** $\log_a 3 - \log_a 2$ **(D)** $\log_a 13$ **(E)** $\log_a 6$ **(F)** NA

SOLUTION. We can, of course, try each one of the potential solutions. For example, if $x = \log_a 3$, then

$$6a^x + 6a^{-x} = 6 \cdot 3 + \frac{6}{3} = 20 \neq 13,$$

so **A** is not the answer. But a better way of proceeding is to set $y = a^x$, then $a^{-x} = 1/y$ and the equation becomes $6y + 6/y = 13$, or $6y^2 - 13y + 6 = 0$. The solutions of this quadratic equation are $y = 3/2$ and $y = 2/3$; i.e., $a^x = 3/2$ or $2/3$. The one with positive x is $3/2$; taking logarithms in base a we see that $x = \log_a 3 - \log_a 2$.

The correct solution is **C.** ■

16. If a, b are real numbers such that $(a+bi)^2 = 3+4i$ and $a < 0$ then b equals:

(A) 1 **(B)** -1 **(C)** 2 **(D)** -2 **(E)** NA

SOLUTION.

$$3 + 4i = (a+bi)^2 = a^2 - b^2 + 2abi,$$

so that $a^2 - b^2 = 3$, $2ab = 4$. Thus $a = 2/b$ and a, b have the same sign. Substituting $a = 2/b$ into $a^2 - b^2 = 3$ we get (after some rearranging) $b^4 + 3b^2 - 4 = 0$; thus $b^2 = 1$ or $b^2 = -4$. But $b^2 = -4$ is impossible, thus $b^2 = 1$, hence $b = \pm 1$. To get $a < 0$ we have to select $b = -1$.

The correct solution is **B.** ■

17. Let x, y be **complex** numbers such that $x \neq 0$, $y \neq 0$, $x + y \neq 0$, and satisfying the equation

$$x^3 + x^2y + xy^2 + y^3 = 0.$$

Determine the value of

$$\left(1 + \frac{x}{y}\right)^{2008} + \left(1 + \frac{y}{x}\right)^{2008}.$$

(A) 2^{1004} **(B)** 2^{1005} **(C)** 2^{2007} **(D)** 2^{2008} **(E)** $2 \cdot 2^{2008}$ **(F)** NA

SOLUTION. We notice first that $x \neq 0 \neq y$ excludes the possibility that $x = y$. Multiplying the equation by $x - y$ gives

$$0 = (x - y)(x^3 + x^2y + xy^2 + y^3) = x^4 - y^4$$

so that $y = \pm x, \pm ix$. But $y = \pm x$ has been excluded, thus $x = \pm iy$ and

$$1 + \frac{x}{y} = 1 \pm i;$$

$$\left(1 + \frac{x}{y}\right)^{2008} = (1 \pm i)^{2008} = (\pm 2i)^{1004} = 2^{1004}.$$

Similarly one sees that

$$\left(1 + \frac{y}{x}\right)^{2008} = 2^{1004}.$$

Thus

$$\left(1 + \frac{x}{y}\right)^{2008} + \left(1 + \frac{y}{x}\right)^{2008} = 2^{1004} + 2^{1004} = 2^{1005}.$$

The correct solution is **B.** ■

18.* What is the smallest that $\log_a b + \log_b a$ can be if $a > 1, b > 1$? Enter your answer directly on the answer sheet.

SOLUTION. Since $a, b > 1$, both $\log_a b$ and $\log_b a$ are positive. Moreover, one is the reciprocal of the other; so if $x = \log_a b$, then $\log_b a = 1/x$ and we are asking what is the smallest possible value of $x + 1/x$ if $x > 0$. It stands to reason that the smallest value is achieved for $x = 1/x$; i.e., for $x = 1$. This would make 2 the smallest value. To assure ourselves that it is indeed so, notice that

$$x + \frac{1}{x} - 2 = \frac{(x-1)^2}{x} \geq 0$$

and is 0 if and only if $x = 1$.

The number to be entered should be **2.** ■

19.* Determine the value of x in degrees such that $0 < x < 36$ and x solves the equation $\sin x + \sin 5x = \cos x + \cos 5x$. Enter your answer directly on the answer sheet.

SOLUTION. Write the equation in the form $\cos x - \sin x = \sin 5x - \cos 5x$ and multiply by $1/\sqrt{2} = \sin 45° = \cos 45°$. By standard trigonometric identities, the equation becomes $\sin(45° - x) = \sin(5x - 45°)$ which will hold for $45 - x$ in the first quadrant if and only if $45 - x = 5x - 45$; i.e., $x = 15$.

The number to be entered should be **15.** ■

20. Determine a so that the identity $\cos^4\theta = a + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$ is valid for all θ.

(A) $\frac{1}{4}$ **(B)** $\frac{1}{8}$ **(C)** $\frac{3}{8}$ **(D)** 1 **(E)** NA

SOLUTION. A quick approach is to just give a value to θ for which all the cosines are known. For example, with $\theta = 0$ the equation becomes $1 = a + (1/2) + (1/8)$, hence $a = 3/8$. The answer must be C or E.

For a more precise approach, by the double angle formula, $\cos^2\theta = (1/2)(1+\cos 2\theta)$; squaring and using the double angle formula again,

$$\cos^4\theta = \frac{1}{4}(1+2\cos 2\theta+\cos^2 2\theta) = \frac{1}{4}\left(1+2\cos 2\theta+\frac{1}{2}(1+\cos 4\theta)\right) = \frac{3}{8}+\frac{1}{2}\cos 2\theta+\frac{1}{8}\cos 4\theta.$$

The correct solution is **C.** ■

21. A quadrilateral has sides of 2, 5, 10 and 11. What is the largest possible area?

(A) 30 **(B)** 36 **(C)** 44 **(D)** 55 **(E)** NA

SOLUTION. To come up with an easy solution of this problem, one should know that of all quadrilaterals with given sides the one that maximizes the area is the *cyclic* one; i.e., the one that can be inscribed in a circle. For such quadrilaterals, the area is given by *Brahmagupta's formula*:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where a, b, c, d are the sides and $s = (a+b+c+d)/2$. In our case $s = (2+5+10+11)/2 = 14$ and

$$A = \sqrt{(14-2)(14-5)(14-10)(14-11)} = \sqrt{1296} = 36.$$

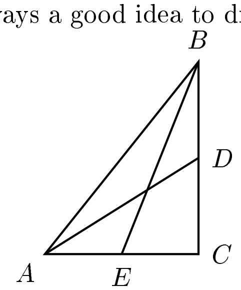
An interesting observation, that works in our case, is that $2^2+11^2 = 125 = 5^2+10^2$. This means that if we put the sides together in the order 2,11,5,10, the quadrilateral will be made up out of two right triangles and it is sort of intuitively clear that that should give the largest area. The area in this configuration works out to $(1/2)(2\times 11)+(1/2)(5\times 10) = 36$; same as before. The correct solution is **B.** ■

22. AB is the hypotenuse of a right triangle ABC. If the median AD is 8 and median BE is 6, then the length of AB is

(A) 5 **(B)** $\sqrt{5}$ **(C)** $2\sqrt{5}$ **(D)** $4\sqrt{5}$ **(E)** 10 **(F)** NA

(The *medians* of a triangle are the segments joining a vertex to the midpoint of the opposite side. In our problem, D is the midpoint of the side BC, E is the midpoint of AC.)

SOLUTION. If no picture is provided, it is always a good idea to draw one.



Let $\alpha = |AE| = |EC|$, $\beta = |BD| = |DC|$. By the Theorem of Pythagoras,

$$|AB|^2 = |AC|^2 + |AD|^2 = 4\alpha^2 + 4\beta^2.$$

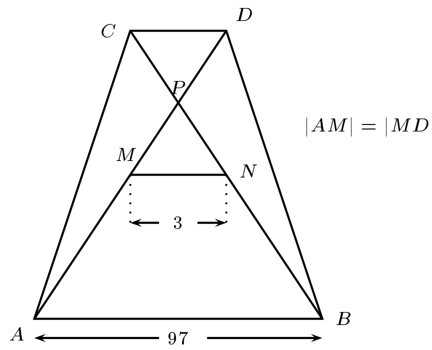
Looking now at the right triangles ADC and BEC, the theorem of Pythagoras gives

$$|AC|^2 + |CD|^2 = |AD|^2 = 64; \quad \text{i.e.,} \quad 4\alpha^2 + \beta^2 = 64,$$
$$|EC|^2 + |CB|^2 = |BE|^2 = 36; \quad \text{i.e.,} \quad \alpha^2 + 4\beta^2 = 36.$$

Adding term by term, $5(\alpha^2+\beta^2) = 100$, thus $\alpha^2+\beta^2 = 20$ and $|AB|^2 = 4(\alpha^2+\beta^2) = 80$. It follows that $|AB| = \sqrt{80} = 4\sqrt{5}$.

The correct solution is **D.** ■

23. The line joining the midpoints of the diagonals of a trapezoid has length 3. If the longer base is 97, determine the length of the shorter base.



(A) 94 **(B)** 92 **(C)** 91 **(D)** 90 **(E)** 89 **(F)** NA

SOLUTION. Some points were added to the original picture, for easy reference. We have three similar triangles:

$$APB \sim MPN \sim CPD$$

We see that $AB/MN = AP/MP$ and that $CD/MN = PD/MP$. Adding,

$$\frac{AB + CD}{MN} = \frac{AD}{MP} = \frac{2AM}{MP}.$$

Now $AM = MD = MP + PD$ so that

$$\frac{2AM}{MP} = 2 + \frac{2PD}{MP} = 2 + \frac{2CD}{MN}.$$

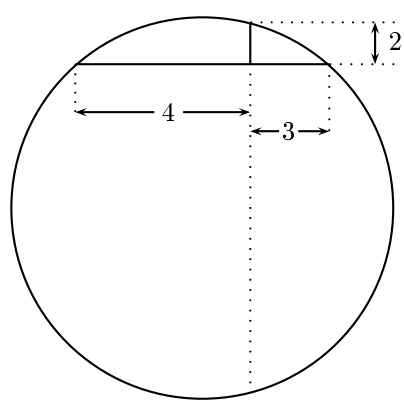
Equating, we get

$$\frac{AB + CD}{MN} = 2 + \frac{2CD}{MN}, \quad \text{hence} \quad CD = AB - 2MN.$$

This proves a general result for trapezoids: *The line joining the midpoints of the diagonals is one half of the* difference *of the two bases.* (Knowing this beforehand would have speeded up finding the solution). In our case, $CD = 97 - 6 = 91$.

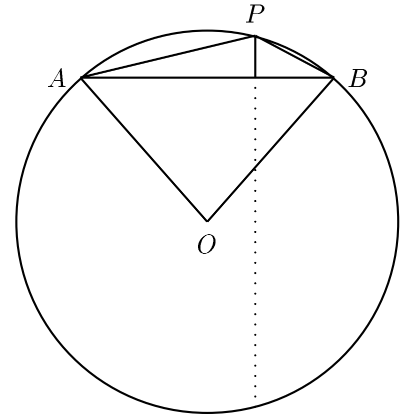
The correct solution is **C.** ■

24. Two perpendicular chords intersect in a circle. The segments of one chord are 3 and 4; one segment of the other has length 2. Determine the diameter of the circle.



(A) $\sqrt{89}$ **(B)** $\sqrt{56}$ **(C)** $\sqrt{61}$ **(D)** $\sqrt{75}$ **(E)** $\sqrt{65}$ **(F)** NA

SOLUTION. Let's add some lines to the picture: Lines from the center O of the circle to the chord AB, and from A and B to P, the point where the other chord segment intersects the circle.



We see that $\angle APB$ is the sum of an angle whose tangent is $3/2$ plus an angle of tangent $4/2 = 2$; thus

$$\tan\angle APB = \frac{2+\frac{3}{2}}{1-2\cdot\frac{3}{2}} = -\frac{7}{4}.$$

A bit of geometry shows that the central angle $\angle AOB$ equals (in degrees) $360 - 2(\angle APB)$ and thus

$$\tan\angle AOB = -\tan(2\angle APB) = -\frac{2\tan\angle APB}{1-(\tan\angle APB)^2} = -\frac{56}{33}.$$

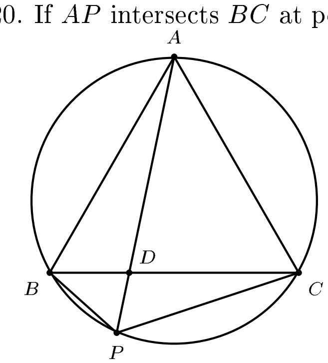
Then $\cos\angle AOB = -\frac{33}{65}$. If $r = |OA| = |OB|$ is the radius of the circle, then

$$7^2 = |AB|^2 = |OA|^2 + |OB|^2 - 2|OA|\,|OB|\cos(\angle AOB) = 2r^2 + 2r^2\frac{33}{65} = \frac{196r^2}{65}.$$

It follows that $r^2 = 65/4$ so that the diameter is $2r = \sqrt{65}$.

The correct solution is **E.** ■

25*. The equilateral triangle ABC is inscribed in a circle. The point P is chosen on the arc BC and the lines AP, BP, and CP are drawn with $PB = 5$ and $PC = 20$. If AP intersects BC at point D, what is the length of AD?



Enter your answer directly on the answer sheet.

SOLUTION. Applying Ptolemy's theorem, we have $AP \cdot BC = AB \cdot CP + AC \cdot BP$. Since ABC is equilateral, $AP = AB + AC = 25$. Note that triangles APC and BPD are similar, since $\angle PAC = \angle PBD$, and $\angle APC = \angle BPD = 60°$. It follows from $AP : PC = BP : PD$ that $25 : 20 = 5 : 25 - AD$, and $AD = 21$

The number to be entered should be **21.** ■