MATH DAY 2006 at FAU

Competition A–Individual

NOTE:

1. The option NA stands for "None of the previous answers is correct."
2. In all questions, i stands for the imaginary unit; $i^2 = -1$.
3. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
4. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1. The degree of $(x^3 - 1)^4 - (x^3 + 2)^4$ as a polynomial in x is

   (A) 12 **(B)** 9 **(C)** 6 **(D)** 3 **(E)** NA

2. The remainder of dividing $x^3 - 5x^2 + 7x + a$ by $x - 1$ is 5. Determine a.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** NA

3. Suppose that $m^2 = n + 2$, $n^2 = m + 2$ $(m \neq n)$. Find the value of $m^3 - 2mn + n^3$.

   (A) 1 **(B)** 0 **(C)** -1 **(D)** -2 **(E)** NA

4. The **minimum** value of $\max\{|x + 1|, |x - 2|\}$ for x real is

   (A) $\frac{2}{3}$ **(B)** 2 **(C)** $\frac{3}{2}$ **(D)** 3 **(E)** NA

5. Suppose x_1 is the smallest and x_2 is the largest solution of $|3x - 2| + |3x + 1| = 3$. Then x_2/x_1 equals

   (A) -3 **(B)** -2 **(C)** -1 **(D)** 2 **(E)** NA

6. Suppose x_1, x_2, x_3 are the three distinct roots of the equation $x^3 - 4x + 2 = 0$. Determine the value of $x_1^2 + x_2^2 + x_3^2$.

   (A) 2 **(B)** 4 **(C)** 8 **(D)** 16 **(E)** NA

7. Let a function $f(x)$ satisfy $f(x + 2) = \frac{1}{f(x)}$ and $f(1) = -5$. Find $f(f(5))$.

   (A) 5 **(B)** $\frac{1}{5}$ **(C)** -5 **(D)** $-\frac{1}{5}$ **(E)** NA

8. Compute $\left(\frac{1+i}{\sqrt{2}}\right)^{2007}$.

   (A) $\frac{1+i}{\sqrt{2}}$ **(B)** $\frac{1-i}{\sqrt{2}}$ **(C)** $\frac{-1+i}{\sqrt{2}}$ **(D)** $\frac{-1-i}{\sqrt{2}}$ **(E)** NA

9. Let z be a complex number such that $|z + 2 - 2i| = 1$. The smallest possible value of $|z - 2 - 2i|$ is

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** NA

10. In how many different ways can 5 be written as a sum of three positive (not necessarily different) integers, if we consider order (that is, two sums will be considered different even if they consist of the same three positive integers, as long as the order of appearance of these integers is different)?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 8 **(E)** NA

11. An 8-inch pizza is cut into 3 equal slices, a 10-inch pizza is cut into 4 equal slices, and a 12-inch pizza is cut into 6 equal slices. If you are on diet but must choose one slice, from which pizza should you choose to get as little as possible?

(A) 8–inch **(B)** 10–inch **(C)** 12–inch **(D)** does not matter **(E)** NA

12. Of 4 tests, each out of 100 marks, my average is 86. What is the lowest possible score I can have on any of these tests?

(A) 0 **(B)** 44 **(C)** 80 **(D)** 85 **(E)** NA

13. When Peter walks to school and rides back home, it takes him $1\frac{1}{2}$ hours. When he rides both ways, it takes only $\frac{1}{2}$ hour. How long would it take him to make the round trip by walking?

(A) $2\dfrac{3}{4}$ hours **(B)** $2\dfrac{1}{2}$ hours **(C)** 2 hours **(D)** $1\dfrac{3}{4}$ hours **(E)** NA

14. What is the greatest common divisor of 21 and 28?

(A) 1 **(B)** 3 **(C)** 4 **(D)** 7 **(E)** NA

15. There are two unequal numbers. When half of the smaller number is subtracted from both, the new larger number is five times the new smaller number. How many times does the old smaller number go into the old larger number?

(A) 2 **(B)** 3 **(C)** 6 **(D)** 8 **(E)** NA

16. Suppose that x_1, x_2, x_3 are three distinct, real solutions of the equation $x^3 + 7x^2 - bx - 8 = 0$, where b is a real constant. Then $\log_2 x_1 + \log_2 x_2 + \log_2 x_3$ is equal to:

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

17. If $2^x = y$, then $\log_4 y$ equals

(A) $2x$ **(B)** $x/2$ **(C)** x **(D)** x^2 **(E)** NA

18. Let x and y be **non-zero complex** numbers satisfying the equation $x^2 + xy + y^2 = 0$. Determine the value of

$$\left(\frac{x}{x+y}\right)^{2007} + \left(\frac{y}{x+y}\right)^{2007}.$$

(A) 2 **(B)** -2 **(C)** 2^{2007} **(D)** -2^{2007} **(E)** NA

19. Suppose that $a_1, a_2, a_3, \ldots$ are in arithmetic progression; $a_1 = \dfrac{1}{3}$, $a_2 + a_5 = 4$, and $a_n = 33$. Determine the value of n.

(A) 47 **(B)** 48 **(C)** 49 **(D)** 50 **(E)** NA

20. $ABCD$ is a square with sides of length 10 inches. How many points are there in the same plane, inside and outside the square, that are equidistant from A and B, and are exactly 6 inches from C?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

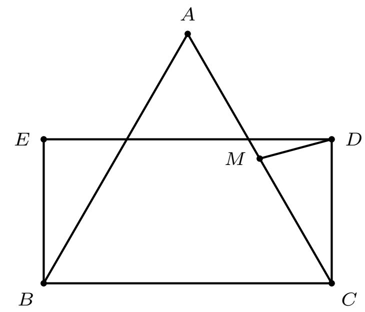
21. Given a semicircle with radius 2. What is the area of the largest rectangle that can be inscribed in the semicircle?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 8 **(E)** NA

22. Which of the following triangles with given sides has the greatest area?

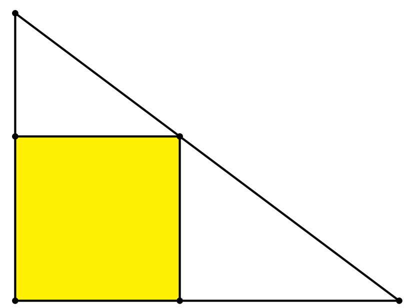
(A) $5, 12, 12$ **(B)** $5, 12, 13$ **(C)** $5, 12, 14$ **(D)** $5, 12, 15$ **(E)** $5, 12, 16$

23. In the diagram below, ABC is an equilateral triangle, and $BCDE$ is a rectangle in which $BC = 2 \cdot CD$. If M is the midpoint of AC, then $\angle CMD =$ __ degrees.



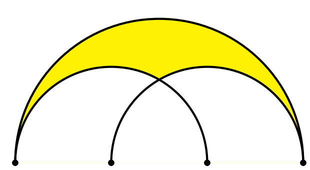
(A) 30 **(B)** 60 **(C)** 75 **(D)** 90 **(E)** NA

24. A square is inscribed in a $3-4-5$ triangle as shown below. What fraction of the triangle does it occupy?



(A) $\frac{12}{25}$ **(B)** $\frac{24}{49}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{25}{49}$ **(E)** NA

25. In the following diagram, the centers of the semicircles are on the same line. If each of the two smaller semicircles has radius 1, find the area of the shaded region.



(A) $\frac{11\pi}{24} - \frac{\sqrt{3}}{4}$ **(B)** $\frac{11\pi}{24} + \frac{\sqrt{3}}{4}$ **(C)** $\frac{11\pi}{12} - \frac{\sqrt{3}}{4}$ **(D)** $\frac{11\pi}{12} + \frac{\sqrt{3}}{4}$ **(E)** NA