MATH DAY 2007 at FAU

Competition B-Teams

SOLUTIONS

NOTE:

1. The option NA stands for "None of the previous answers is correct."
2. In all questions, i stands for the imaginary unit; $i^2 = -1$.
3. $\log_b a$ denotes the logarithm in base b of a; $\log_b a = c$ if and only if $b^c = a$.
4. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.

THE QUESTIONS

1. Let a be a real number. Given that the coefficient of x^7 in the expansion of $(x+a)^{10}$ is 15, determine the value of a.

   (A) $\frac{1}{8}$ **(B)** $\frac{1}{4}$ **(C)** $\frac{1}{2}$ **(D)** 1 **(E)** NA

   SOLUTION. The coefficient of x^7 is $\binom{10}{7} a^3 = 120a^3$. Equating to 15 we get $a^3 = 1/8$, thus $a = 1/2$.

   The correct solution is **C.**

---

2. The polynomial $x^7 - 5x^6 + 3x^2 + 1$ is divided by the polynomial $x - 1$. The remainder is

   (A) $x - 1$ **(B)** 1 **(C)** 0 **(D)** 3 **(E)** NA

   SOLUTION. The remainder of dividing a polynomial in x by $x - a$ equals the value of the polynomial for $x = a$. We have $1^7 - 5 \cdot 1^6 + 3 \cdot 1^2 + 1 = 0$.

   The correct solution is **C.**

---

3. Determine the minimum value of $x_1^2 + x_2^2$ if x_1, x_2 are the two real roots of an equation $x^2 - 2mx - (2m+2) = 0$ for some real number m. To be quite specific, the values of x_1, x_2 change as m changes, presumably for some m the sum of their squares will be smallest.

   (A) 3 **(B)** 2 **(C)** 1 **(D)** 0 **(E)** NA

   SOLUTION. The equation has distinct real roots for all values of m. We have $x_1 + x_2 = 2m$, $x_1 x_2 = -(2m+2)$, so that

   $$x_1^2 + x_2^2 = (x_1 + x_2)^2 - 2x_1 x_2 = (2m)^2 + 2(2m+2) = 4m^2 + 4m + 4 = (2m+1)^2 + 3.$$

   Since $(2m+1)^2 \geq 0$ the smallest value is achieved for $m = -1/2$.

   The correct solution is **A.**

---

4. Let $f(x) = x^2 + 4x + 3$ and let a, b, c be constants such that $f(ax + b) = x^2 + 10x + c$. Determine c.

(A) 6 **(B)** 12 **(C)** 24 **(D)** 48 **(E)** NA

SOLUTION. One can see at once that $a = \pm 1$. If $a = 1$, then

$$x^2 + 10x + c = (x + b)^2 + 4(x + b) + 3 = x^2 + (2b + 4)x + (b^2 + 4b + 3)$$

and $2b + 4 = 10$, $c = b^2 + 4b + 3$ giving $b = 3$, $c = 24$. If $a = -1$, then

$$x^2 + 10x + c = (-x + b)^2 + 4(-x + b) + 3 = x^2 - (2b + 4)x + (b^2 + 4b + 3)$$

so that $-2b - 4 = 10$, $c = b^2 + 4b + 3$. This time $b = -7$, but c again equals 24.

The correct solution is **C.**

---

5. The four real roots of $(x^2 - 2x + m)(x^2 - 2x + n) = 0$ form an arithmetic progression with the first root being $\frac{1}{4}$. Determine $|m - n|$

(A) 1 **(B)** $\frac{3}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{3}{8}$ **(E)** NA

SOLUTION. Let r be the ratio of the arithmetic progression so that the roots are $1/4$, $(1/4)+r$, $(1/4)+2r$, $(1/4)+3r$. We can assume $m > n$. Then $1/4, (1/4) + 3r$ will be roots of $x^2 - 2 + n = 0$ while $(1/4) + r, (1/4) + 2r$ are the roots of $x^2 - 2 + m = 0$. Thus

$$\frac{1}{2} + 3r = \frac{1}{4} + \left(\frac{1}{4} + 3r\right) = 2$$

from which we get $r = 1/2$. The roots in sequence are $1/4$, $3/4$, $5/4$, $7/4$ and

$$m - n = \frac{3}{4} \cdot \frac{5}{4} - \frac{1}{4} \cdot \frac{7}{4} = \frac{1}{2}.$$

The correct solution is **C.**

---

6. Let $f(x)$ be a function such that for all integers x and y,

$$f(x + y) = f(x) + f(y) + 6xy + 1 \qquad \text{and} \quad f(-x) = f(x).$$

Determine $f(3)$.

(A) 26 **(B)** 27 **(C)** 52 **(D)** 53 **(E)** NA

SOLUTION. If $x = y = 0$, the equation for f becomes $f(0) = 2f(0) + 1$, thus $f(0) = -1$. Taking now $y = -x$ we see that

$$-1 = f(0) = f(x - x) = f(x) + f(-x) - 6x^2 + 1 = 2f(x) - 6x^2 + 1$$

giving $f(x) = 3x^2 - 1$. For $x = 3$, $f(3) = 26$.

The correct solution is **A.**

---

7. If $4x+3y+7z=12$, $9x-2y-3z=8$, and $6x+4y-z=9$, how much is $x-7y+6z$?

(A) 2 **(B)** 4 **(C)** 6 **(D)** 10 **(E)** NA

SOLUTION. The quickest way of doing this is if one spots that the sum of the x coefficients of the first two equations minus twice the x coefficient of the third equation is 1, the x coefficient of the expression whose value one has to determine. This may lead one to notice that the same pattern holds for all coefficients:

$$x-7y+6z=(4x+3y+7z)+(9x-2y-3z)-2(6x+4y-z)=12+8-2\cdot 9=2.$$

The correct solution is **A.**

---

8. Compute $\left(\frac{1-i\sqrt{3}}{2}\right)^{2007}$.

(A) 1 **(B)** -1 **(C)** $\frac{1+i\sqrt{3}}{2}$ **(D)** $\frac{1-i\sqrt{3}}{2}$ **(E)** NA

SOLUTION. We have $\frac{1-i\sqrt{3}}{2}=\cos 60^\circ - i\sin 60^\circ$; by De Moivre's formula

$$\left(\frac{1-i\sqrt{3}}{2}\right)^{3}=\cos 180^\circ - i\sin 180^\circ = -1.$$

Since 2007 is an odd multiple of 3 it follows that $\left(\frac{1-i\sqrt{3}}{2}\right)^{2007}=-1.$

The correct solution is **B.**

---

9. How many positive integers not exceeding 120 are divisible by 3, 4, **or** 5?

(A) 60 **(B)** 72 **(C)** 96 **(D)** 120 **(E)** NA

Note: "or" is inclusive. For example 15, 20 and 60 are integers divisible by 3, 4, or 5.

SOLUTION. Every third integer is a multiple of 3, since $120=40\cdot 3$, there are 40 multiples of 3 in the list. Similarly, every fourth integer is a multiple of 4; since $120=30\cdot 4$, there are 30 multiples of 4. However, every third one of these multiples is also a multiple of 3, leaving only 20 new numbers to consider. There are 24 multiples of 5, of these 8 are also multiples of 3, 6 are also multiples of 4, and 2 (60 and 120) are multiples of both 3 and 4. This leaves $24-(8+6-2)=12$ multiples of 5 that are neither multiples of 3 or 4. The number in question is $40+20+12=72$.

The correct solution is **B.**

---

10. The number 5 can be written as a sum of the squares of two positive integers in exactly one way: $5=1+4$. In how many ways can 325 be written as the sum of the squares of two positive integers?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

SOLUTION. One approach is to simply subtract squares from 325 until we recognize a square. The smaller one of the squares adding up to 325 has to be less that $\sqrt{325/2} \approx 12.7$ so that we can stop at 12^2. In this way we see that

$$325 = 1^2 + 18^2 = 6^2 + 17^2 = 10^2 + 15^2$$

and no others.

The correct solution is **C.**

---

11. A sequence of numbers $a_0, a_1, a_2, \ldots$ is given by $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} - a_{n-2}$ when $n \geq 2$. Determine a_{500}

(A) 0 **(B)** 1 **(C)** -1 **(D)** 500 **(E)** 250 **(F)** NA

SOLUTION. One sees that the terms of the sequence form the pattern $0, 1, 1, 0, -1, -1$ repeated indefinitely. That is, if n is a multiple of 6, then a_n through a_{n+5} are given by $0, 1, 1, 0, -1, -1$. Since $500 = 498 + 2$, and 498 is a multiple of 6, we have $a_{498} = 0$, $a_{499} = 1$, $a_{500} = 1$.

The correct solution is **B.**

---

12. Consider the sequence

$$2, 3, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 17, \ldots$$

obtained from the sequence $\{1, 2, 3, 4, \ldots\}$ of all positive integers by omitting all squares. What is the value of the 2007-th number in this sequence?

(A) 2049 **(B)** 2050 **(C)** 2051 **(D)** 2052 **(E)** NA

SOLUTION. Let n be the 2007-th term in the sequence. If m is such that $m^2 < n < (m+1)^2$, then there are exactly m squares preceding n meaning that n has been shifted m positions forward as the squares were removed, in other words $n - m = 2007$ or $n = 2007 + m$. Thus $m^2 < 2007 + m < (m+1)^2 + 1$ and since we are dealing with integers this can be rewritten as

$$m(m-1) < 2007 \leq m(m+1).$$

Then $m = 45$ and $n = 2052$.

The correct solution is **D.**

---

13. A teacher takes a survey of her class and finds that of her students, 25 have taken Geography, 17 have taken American History, and 21 have taken Spanish. Of the 25 students who took Geography, 10 also took Spanish and 7 of these 25 students who took Geography also took American History. There were 8 students who took American History and Spanish. There were 3 students who took all three courses. Assuming that each student in the class took at least one of these three courses, how many students are there in all?

(A) 25 **(B)** 38 **(C)** 41 **(D)** 63 **(E)** NA

SOLUTION. Let G be the set of students who took Geography, A the set of students who took American History, and S the set of students who took Spanish. In general, if X is a set, denote by $|X|$ the number of elements in the set. Then

$$\begin{aligned} |G \cup A \cup S| &= (|G| + |A| + |S|) - (|G \cap A| + |G \cap S| + |A \cap S|) + |G \cap A \cap S| \\ &= (25 + 17 + 21) - (10 + 7 + 8) + 3 = 41 \end{aligned}$$

The correct solution is **C.**

14. On a 12-hour clock the time now is 11 o'clock. What time will it be 100 hours from now?

(A) 3 **(B)** 4 **(C)** 7 **(D)** 8 **(E)** NA

SOLUTION. $100 = 8 \cdot 12 + 4$, $11 + 4 = 15$, the time will be $15 - 12 = 3$ o'clock.

The correct solution is **A.**

---

15. How many 2-digit numbers are there which are squares and the sum of whose digits is also a square?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

SOLUTION. The only two digit squares are 16, 25, 36, 49, 64, 81. Of these, only 36 and 81 have digits adding up to a square, namely 9 in both cases.

The correct solution is **B.**

---

16. Arrange the four numbers

$$x = 2^{8 \text{ million}}, \qquad y = 3^{6 \text{ million}}, \qquad z = 5^{4 \text{ million}}, \qquad w = 6^{2 \text{ million}}$$

in *descending* order.

(A) x, y, z, w **(B)** y, z, w, x **(C)** w, x, z, y **(D)** y, z, x, w **(E)** NA

SOLUTION. We can also write these numbers in the form

$$x = 256^{1 \text{ million}}, \qquad y = 729^{1 \text{ million}}, \qquad z = 625^{1 \text{ million}}, \qquad w = 36^{1 \text{ million}}$$

from which we see that the correct (descending) order is y, z, x, w.

The correct solution is **D.**

---

17. The numbers 1 to 20 are arranged in such a way that the sum of each pair of adjacent numbers is a prime:

$$20,\ p,\ 16,\ 15,\ 4,\ q,\ 12,\ r,\ 10,\ 7,\ 6,\ s,\ 2,\ 17,\ 14,\ 9,\ 8,\ 5,\ 18,\ t.$$

What is the number s?

(A) 1 **(B)** 3 **(C)** 11 **(D)** 13 **(E)** 19

SOLUTION. A first analysis reveals that we must have $p = 3$, $q = 1$ or 19, $r = 1$ or 19 and $s = 1$ or 11. A second analysis shows that since $1, 19$ are preempted by q and r, s has to be 11.

The correct solution is **C.**

---

18. Let $f(x) = \log_a x$, where $0 < a < 1$. Suppose that the ratio between the largest and smallest value of the function $f(x)$ over the interval $[a, 2a]$ is 3. Determine the value of a.

(A) $\frac{1}{4}$ **(B)** $\frac{\sqrt{2}}{4}$ **(C)** $\frac{1}{2}$ **(D)** $\frac{\sqrt{2}}{2}$ **(E)** NA

SOLUTION. Since $0 < a < 1$, $\log_a$ decreases and the largest value will be $\log_a a = 1$, the smallest $\log_a(2a) = \log_a 2 + 1$. Thus

$$3 = \frac{1}{1 + \log_a 2} \quad \text{giving} \quad \log_a 2 = -\frac{2}{3},$$

hence $a^{-2/3} = 2$, so $a = 1/\sqrt{8} = 1/(2\sqrt{2})$.

The correct solution is **B.**

---

19. If x is a real number and $3^x - 3^{-x} = 3$, then 9^x equals

(A) 1 **(B)** $(11 + 3\sqrt{13})/2$ **(C)** $(11 - 3\sqrt{13})/2$ **(D)** $(3 + \sqrt{13})/2$ **(E)** NA

SOLUTION. Let $y = 3^x$. Then the equation becomes $y - y^{-1} = 3$; multiplying by y, $y^2 - 1 = 3y$, thus $y^2 - 3y - 1 = 0$. Solving, $y = (3 \pm \sqrt{13})/2$. But $y = 3^x > 0$, thus $3^x = (3 + \sqrt{13})/2$. Now

$$9^x = (3^x)^2 = \left(\frac{3 + \sqrt{13}}{2}\right)^2 = \frac{11 + 3\sqrt{13}}{2}.$$

The correct solution is **B.**

---

20. A convex polygon has 35 diagonals. How many sides does it have?

(A) 8 **(B)** 10 **(C)** 20 **(D)** 35 **(E)** NA

SOLUTION. If the polygon has n sides it also has n vertices. Each (unordered) pair of vertices determines either a side or a diagonal. There are $\binom{n}{2} = n(n-1)/2$ such pairs, thus

$$35 = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

We see that $n = 10$.

The correct solution is **B.**

---

21. The sum of the lengths of all the edges of a cube is K inches. If the total surface area of the cube has the same numerical value K square inches, what is its volume in cubic inches?

(A) 1 **(B)** 2 **(C)** 8 **(D)** K^3 **(E)** NA

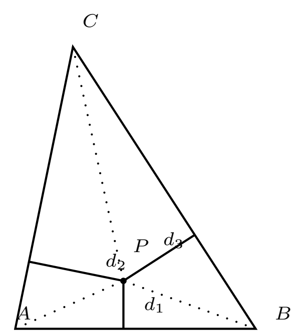
SOLUTION. Let ℓ be the length of an edge of the cube. A cube has 12 edges, thus $12\ell = K$. It has 6 faces, thus $6\ell^2 = K$. Equating we get $\ell = 2$ giving a volume of $2^3 = 8$ cubic inches.

The correct solution is **C.**

22. A point P is in the **interior** of a triangle whose sides are 5 inches, 6 inches and 7 inches long. If the distance from P to the side that is 5 inches long is 2 inches, and the distance from P to the 6 inch long side is 3 inches, how far is P from the 7 inch side?

(A) $\frac{12\sqrt{6}-28}{7}$ **(B)** $\frac{12\sqrt{6}-27}{7}$ **(C)** $\frac{6\sqrt{3}-9}{7}$ **(D)** $\frac{6\sqrt{3}-8}{7}$ **(E)** NA

SOLUTION. Label the vertices of the triangle by A, B, C so that $AB = 5$, $AC = 6$ and $BC = 7$. The picture below, **not to scale**, shows the point P inside the triangle ABC; the perpendiculars from P to the side are drawn in; their lengths d_1, d_2, d_3 are the distances from P to the respective sides.



We are given that $d_1 = 2$, $d_2 = 3$ and d_3 is to be determined. Now d_1 is also the height of triangle APB, so that the area of APB is $\frac{1}{2}(AB)d_1 = 5$. Similarly the area of APC is $\frac{1}{2}(AC)d_2 = 9$ while the area of BPC is $\frac{1}{2}(BC)d_3 = (7d_3)/2$. Thus

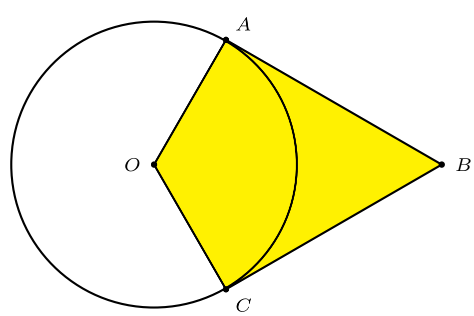
$$\text{Area } ABC = \text{Area } APB + \text{ Area } APC + \text{Area } BPC = 14 + \frac{7d_3}{2} = \frac{28+7d_3}{2}.$$

By Heron's formula, the area of the triangle ABC is $\sqrt{9 \cdot 4 \cdot 3 \cdot 2} = 6\sqrt{6}$. Equating the two expressions for the area gives

$$6\sqrt{6} = \frac{28+7d_3}{2}, \qquad \text{thus } d_3 = \frac{12\sqrt{6}-28}{7}.$$

The correct solution is **A.**

23. In the following diagram, AB and CB are tangent to the circle, and triangle ABC is equilateral. If the radius of the circle is 1, what is the area of the quadrilateral $OABC$?



(A) $\sqrt{3}$ **(B)** $2\sqrt{3}$ **(C)** 3 **(D)** π **(E)** NA

SOLUTION. By the law of sines applied to the triangle OAB, noticing that $\angle OAB$ is a right angle and $\angle OBA = 30°$, hence $\angle AOB = 60°$,

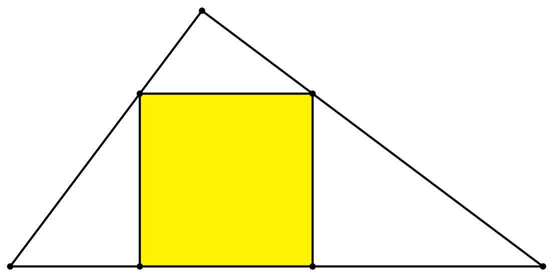
$$\frac{\sin 60°}{AB} = \frac{\sin 30°}{OA}; \quad i.e., \frac{\sqrt{3}}{2AB} = \frac{1}{2},$$

hence $AB = \sqrt{3}$. The area of triangle ABC is $\frac{1}{2} OA \cdot AB = \sqrt{3}/2$; the area of the quadrilateral is twice this amount.

The correct solution is **A.**

---

24. The following figure is built by using little rods all of equal lengths. If the shape of the large triangle is $3:4:5$ and the shaded portion is a square, what is the least amount of little rods to make up the figure?



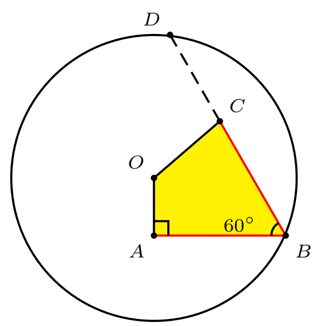
(A) 120 **(B)** 242 **(C)** 624 **(D)** 846 **(E)** NA

SOLUTION. By similarity of triangles (or otherwise) one sees that the length of the sides of the square is $60/37$. The length of the rods should be the **largest** number u that goes in an even number of times in $3, 4, 5$ and $37/60$ and (because 37 is prime) it is easy to see that one cannot do better than $u = 1/37$. With this length we need $3 \cdot 37 = 111$, $4 \cdot 37 = 148$, $5 \cdot 37 = 185$ rods, respectively, for the three sides of the triangle. Each side of the square needs 60 rods to be built, but one side is part of the base of the triangle. We thus need only an additional $3 \cdot 60 = 180$ rods to complete the square. Adding up, we need (at least) $111 + 148 + 185 + 180 = 624$ rods.

The correct solution is **C.**

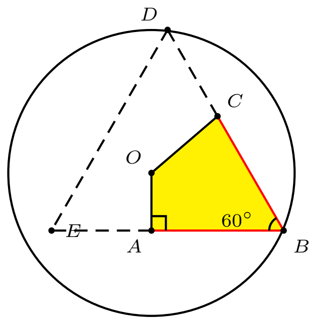
---

25. If $AB = BC$ and $OA = 1$, find the length of CD.



(A) 1 **(B)** $\sqrt{2}$ **(C)** $\sqrt{3}$ **(D)** 2 **(E)** NA

SOLUTION. Extend BA to E such that $AE = CD$. Then $BE = BA + AE = BC + CD = BD$. Since $\angle B = 60°$, triangle BDE is equilateral. Now, O lies on the perpendicular bisector of BD, which passes through E (given an equilateral triangle with two vertices on a circle, the line through the third vertex and the center of the circle is the perpendicular bisector of the side determined by the two vertices on the circle).



It follows that $\angle OEA = 30^\circ$, and $CD = AE = OA \cdot \cot 30^\circ = \sqrt{3}$.

The correct solution is **C.**