Quantum Collision-Finding in Non-Uniform Random Functions

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Motivation

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$$H : [M] \to [N]$$

be a hash function.

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The *collision resistance* of H is a measure of how difficult it is to find $x, y \in [M]$ such that $H(x) = H(y)$.

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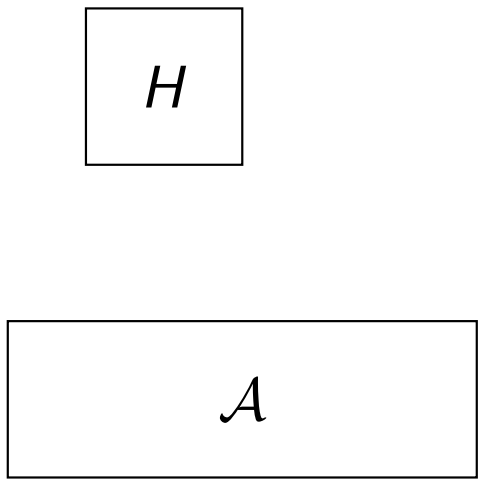
We can allow a *quantum* query to H by

$$U_H : \sum_{\substack{x\in[M]\\ y\in[N]}} \alpha_{x,y}|x\rangle|y\rangle \mapsto \sum_{\substack{x\in[M]\\ y\in[N]}} \alpha_{x,y}|x\rangle|y \oplus H(x)\rangle$$

Generic Security



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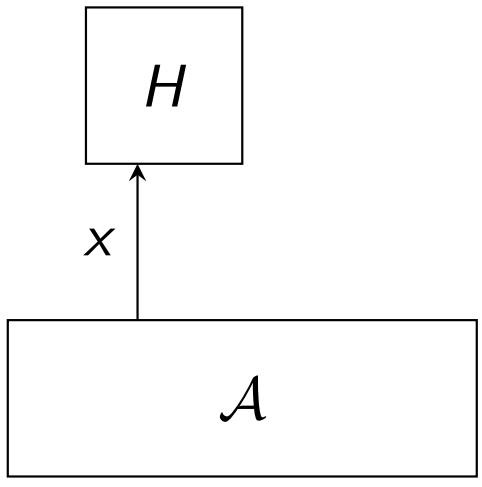

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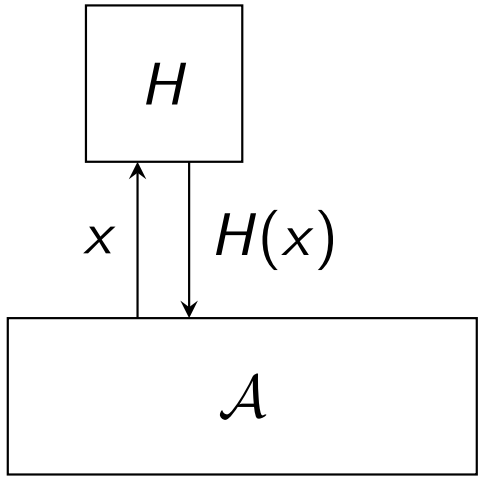

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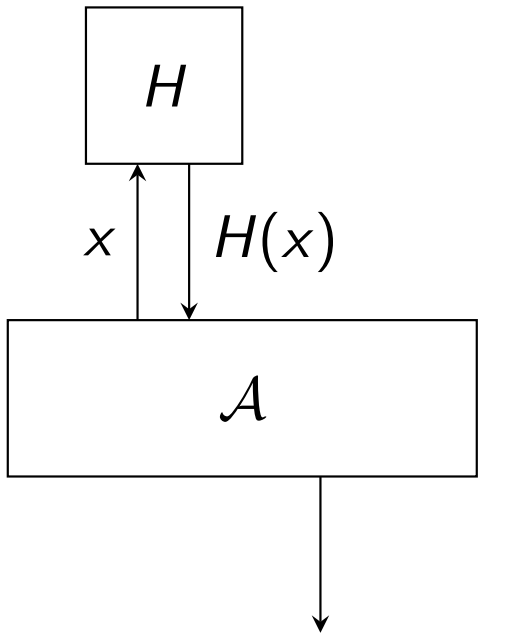
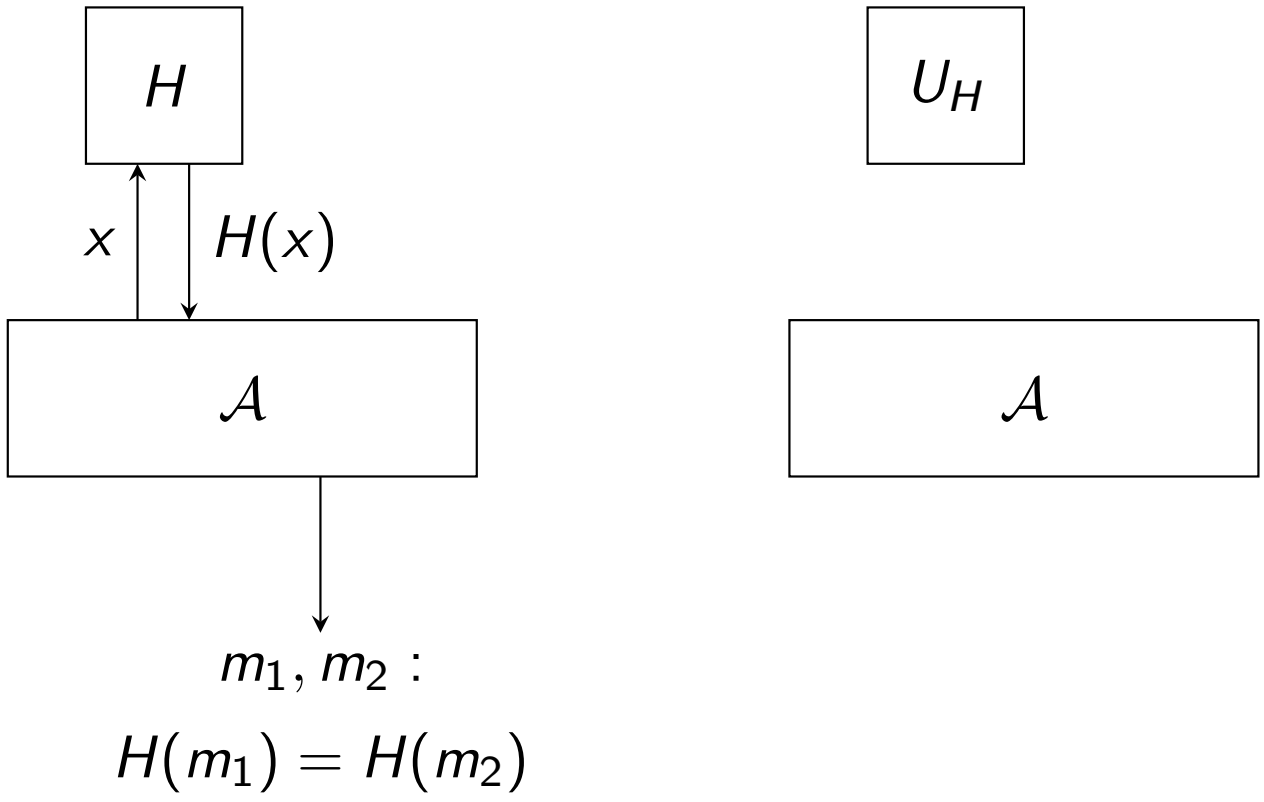


m_1, m_2 :

$H(m_1) = H(m_2)$

Generic Security



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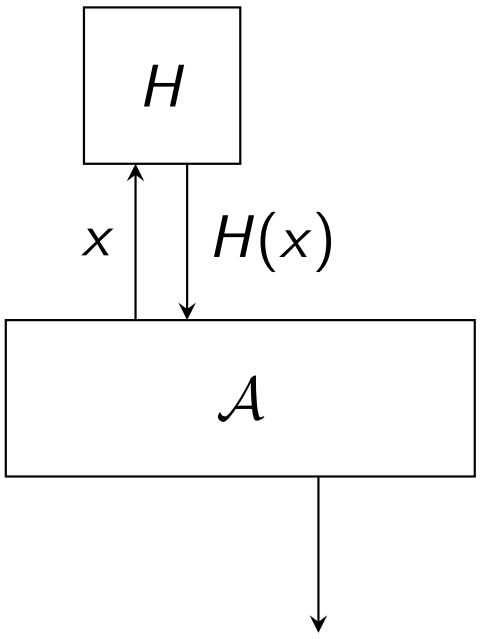

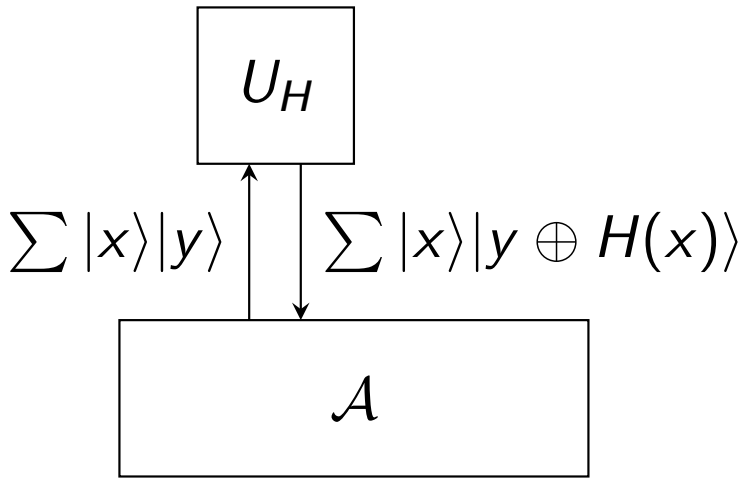

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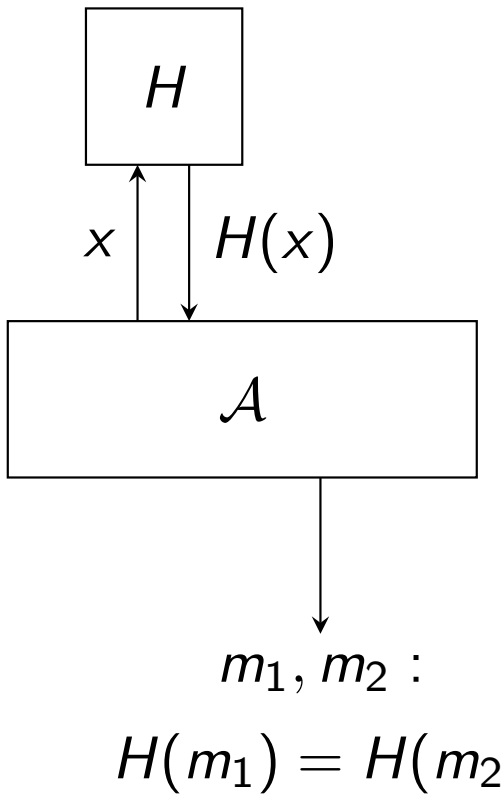

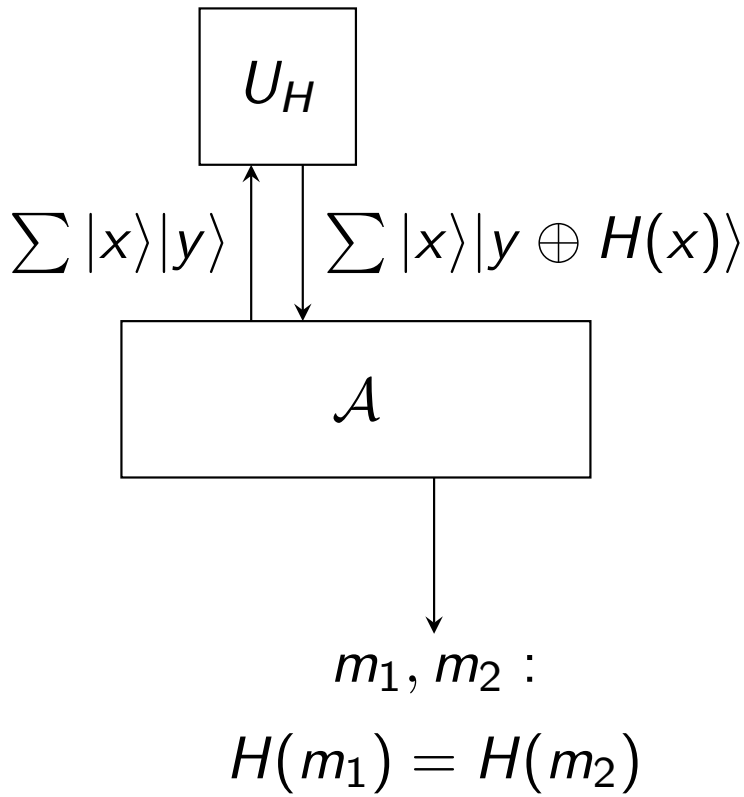

Collision Resistance

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- There is an algorithm that finds a collision in H (with constant probability) and makes $O(N^{1/3})$ queries to U_H.

Results from "A Note on the Quantum Collision and Set Equality Problems" by Mark Zhandry (2015).

Motivation

When H is uniform, the query complexity is $\Theta(N^{1/3})$.

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Is only considering *uniform* functions *enough?*

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- Uniformity is a very strong condition on a function — considering non-uniform can relax our security assumptions.
- Some crypto functions are certainly *not* uniform, e.g., if H_1, H_2 are uniform then $H_1 \circ H_2$ is not.
- Proofs of Fujisaki-Okamoto require collision resistance of non-uniform functions ($f \circ H$).

Definitions

Let D be a distribution on $[N]$. Then we say that D has min-entropy k if

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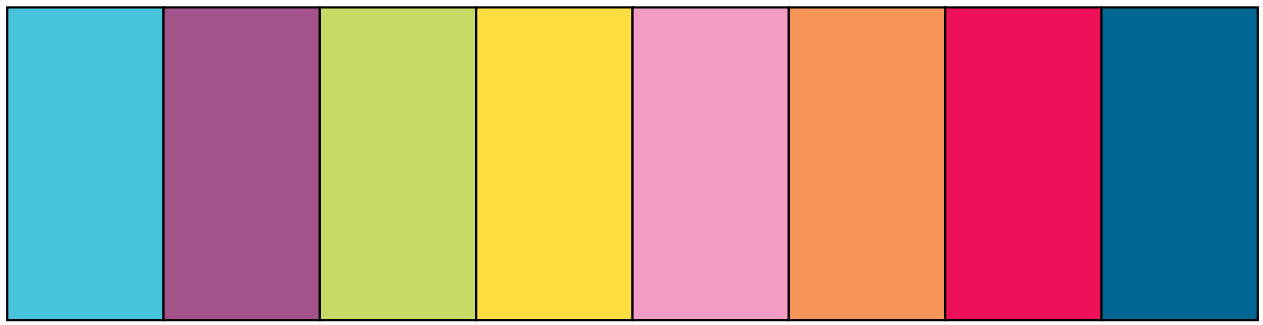
We say that a function H has distribution D if $H(x)$ has distribution D for all $x \in [M]$, and all are independent.

Examples

$N = 8$, $D_1 =$ uniform / flat:



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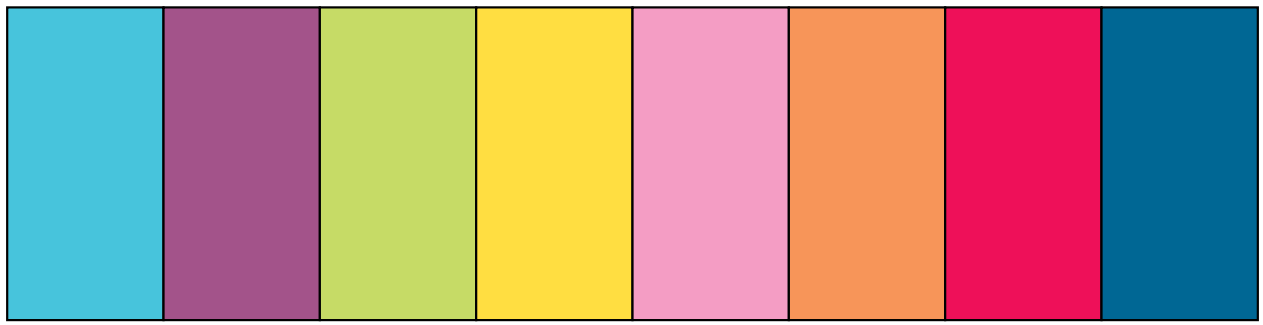
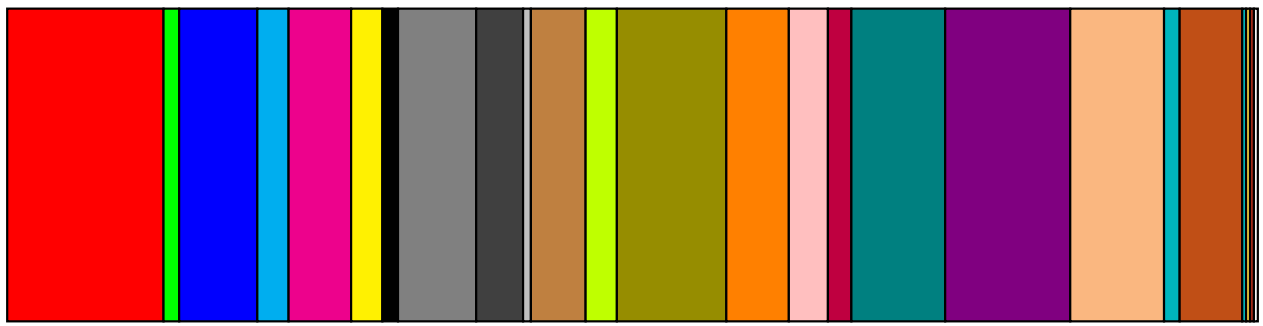


$N = 25$, $D_2 =$ *generic* :



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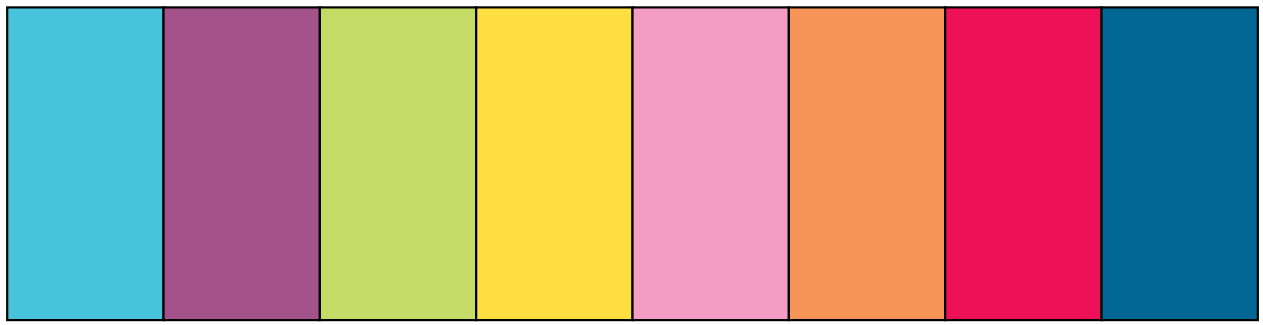

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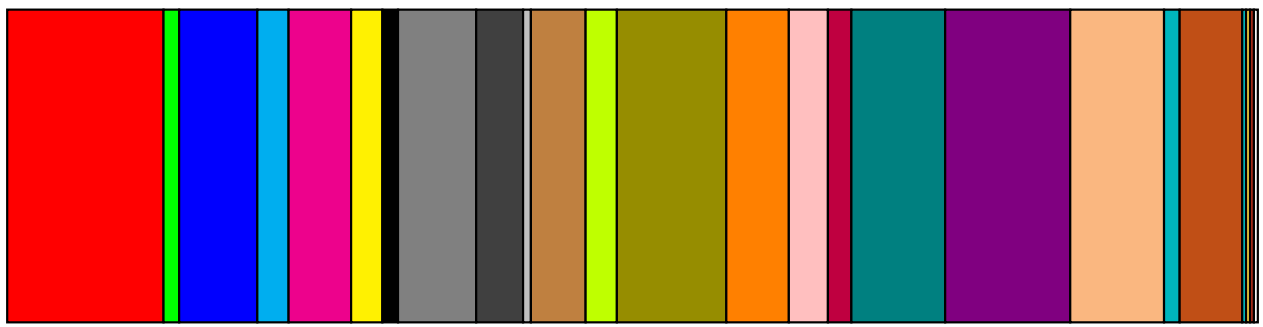
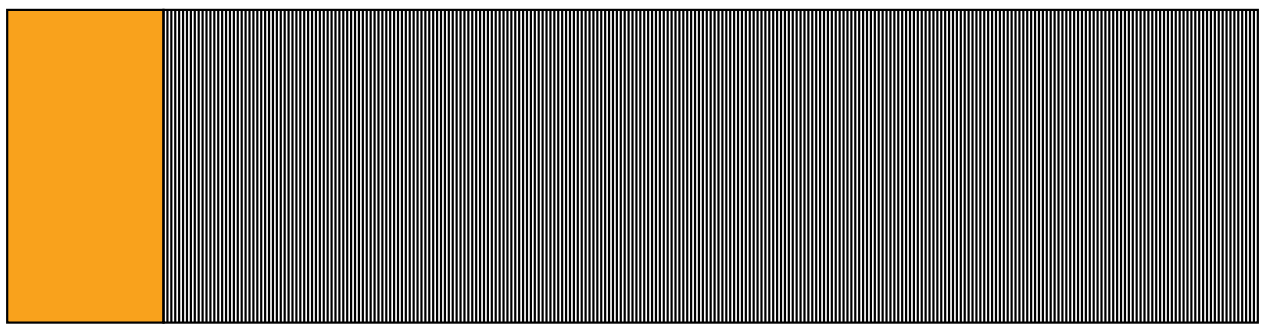


Both have min-entropy 3

Examples

$N = \Omega(M)$, $D_3 =$ delta:



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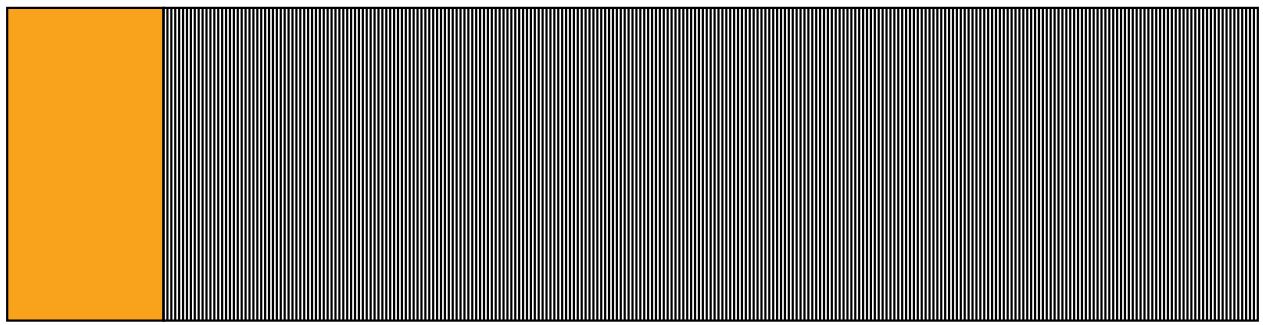

Still min-entropy 3

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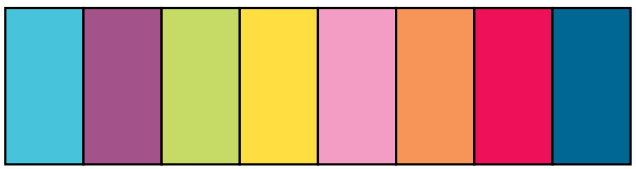
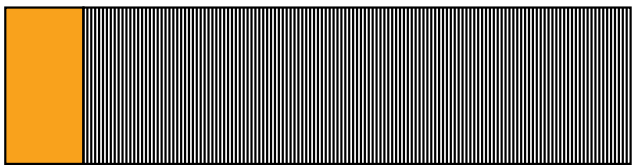
$\beta\Big($  $\Big) = 2^k$

$\beta\Big($  $\Big) \approx 2^{2k}$

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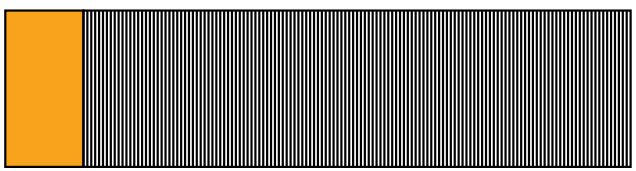
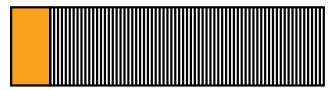
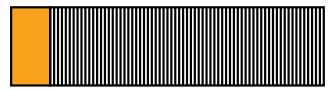
$\beta\Big($  $\Big) \approx 2^{2k}$

$\beta\Big($  $\Big) \in [2^k, 2^{2k})$

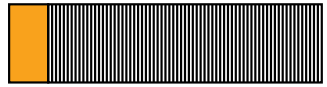
Previous Work

To find a collision with constant probability...			
it takes at **least** this many queries	$2^{k/3}$	?	$2^{k/9}$
it **can** be done in this many queries	$2^{k/3}$	?	?

Independent Work — Ebrahimi & Unruh

To find a collision with constant probability...			
it takes at **least** this many queries	$2^{k/3}$	$2^{k/2}$	$2^{k/5}$
it **can** be done in this many queries	$2^{k/3}$	$2^{k/2}$	$\beta^{1/3}$

Our Work

To find a collision with constant probability...			
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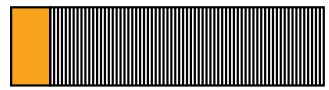
To find a collision with constant probability...			
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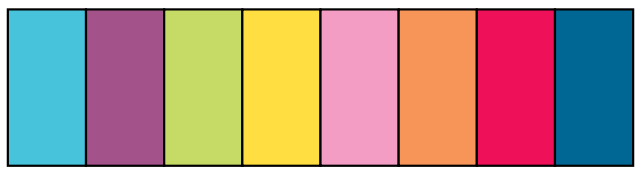
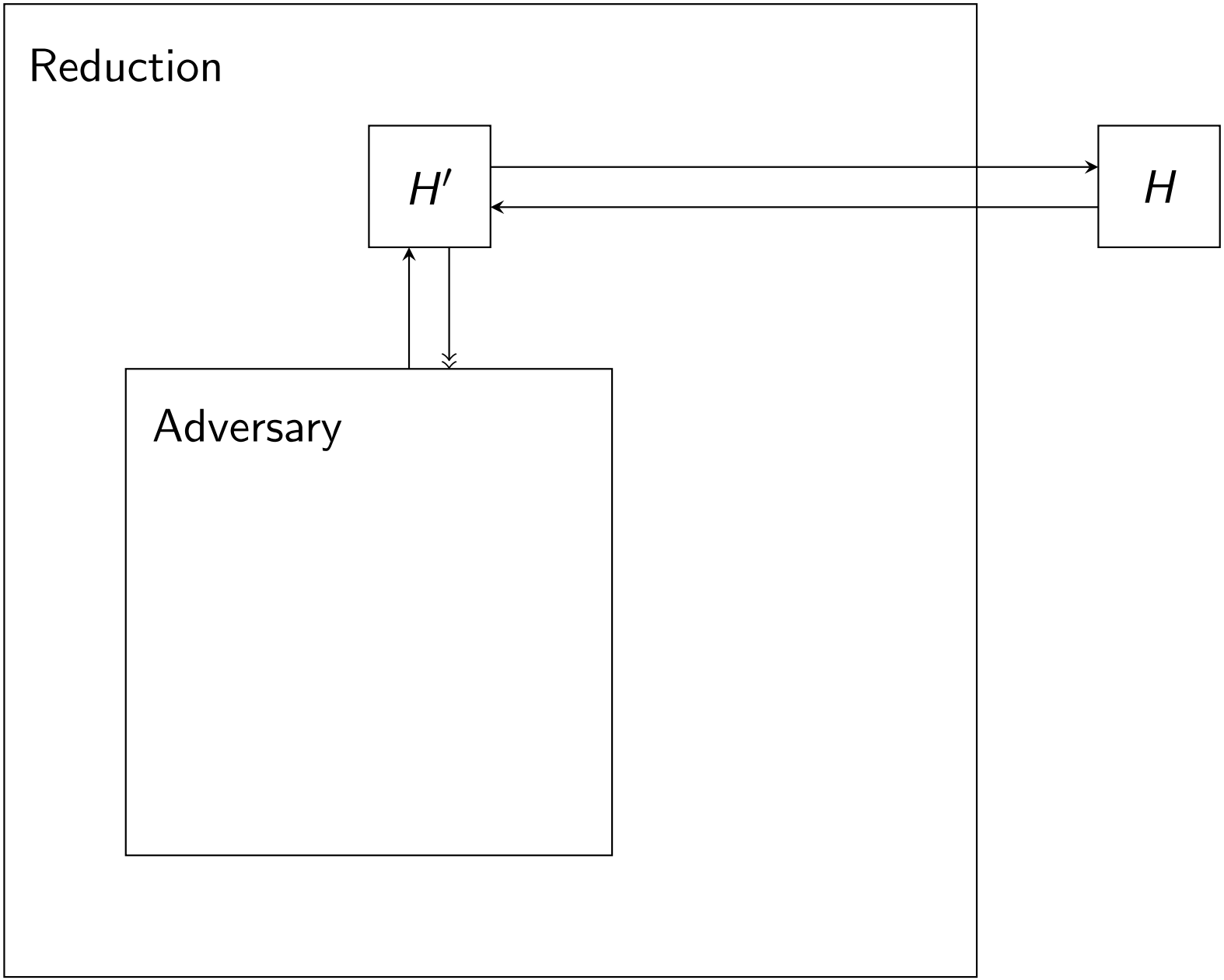
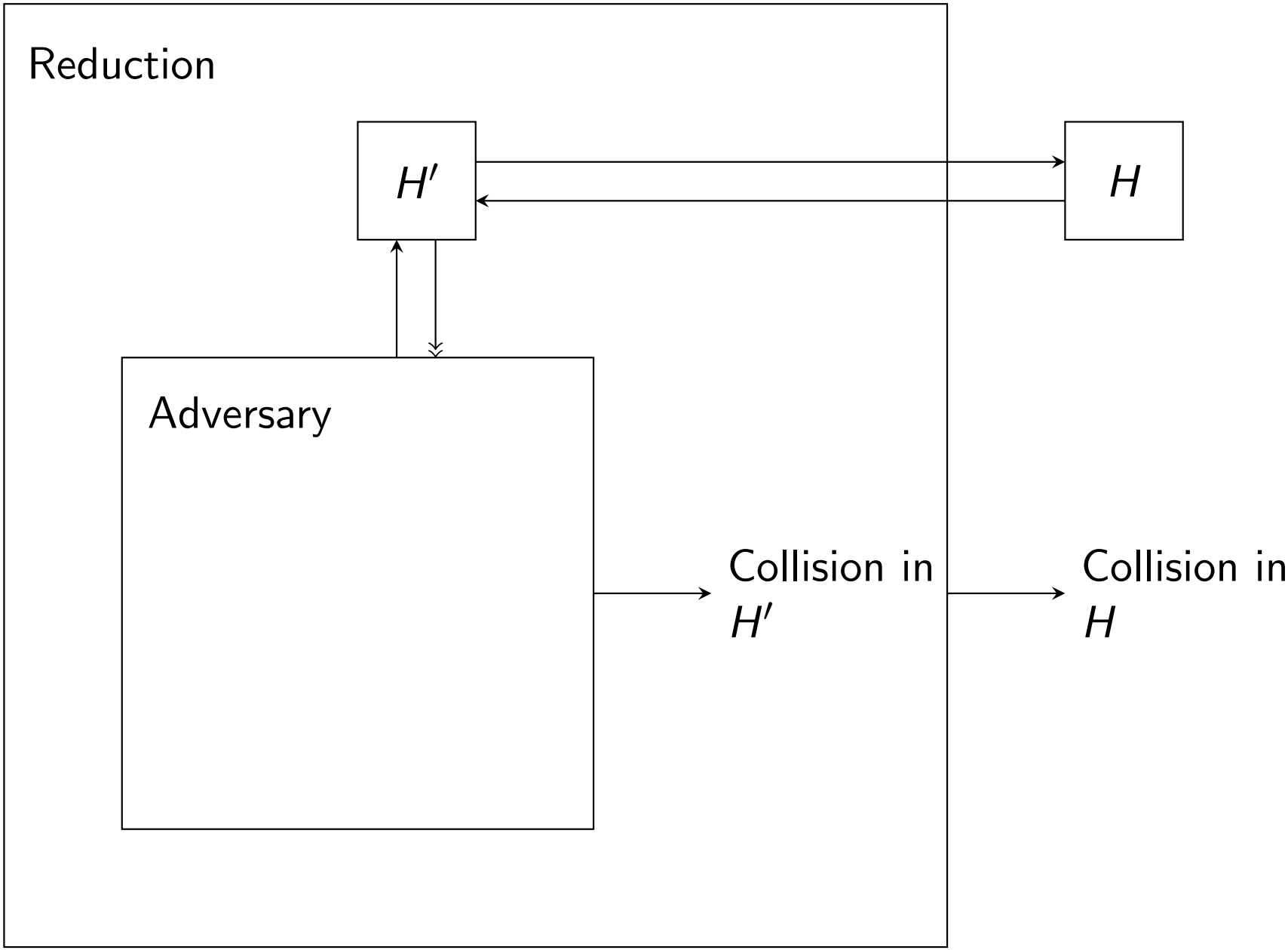
in q queries to $U_{H'}$, with probability p, can be used to find a collision in a hash function H with outputs distributed by



in $2q$ queries to U_H, with probability $p/2$.


Reduction
H'
H
Adversary


Reduction
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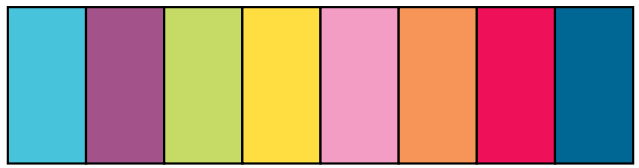
Collision in H'
Collision in H

Proof outline

Idea: Use a distribution conversion to ‘chop up’



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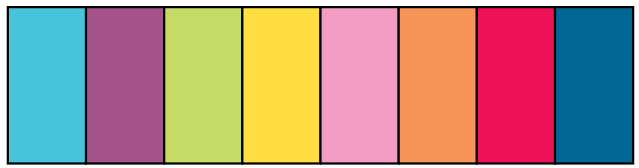

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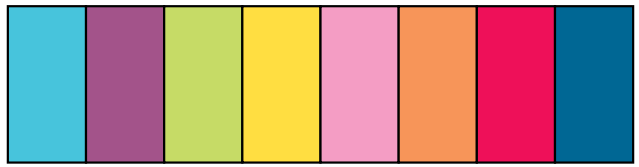
Then show that a collision in the generic distribution should imply a collision in the uniform!

Simulating H'

Say we have H with output distribution



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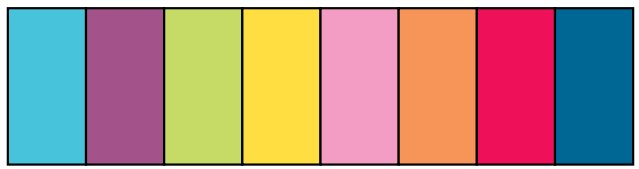
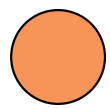


We pick $m \xleftarrow{\$} [M]$ and compute

$$H(m) = \quad$$ 

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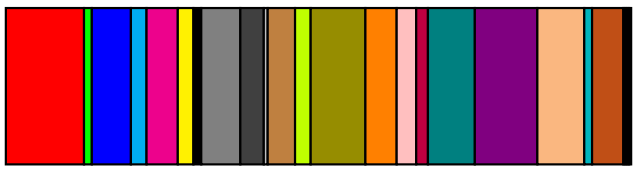

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So when we compute $H(m) =$  , we will choose what $H'(m)$ can be based on this.

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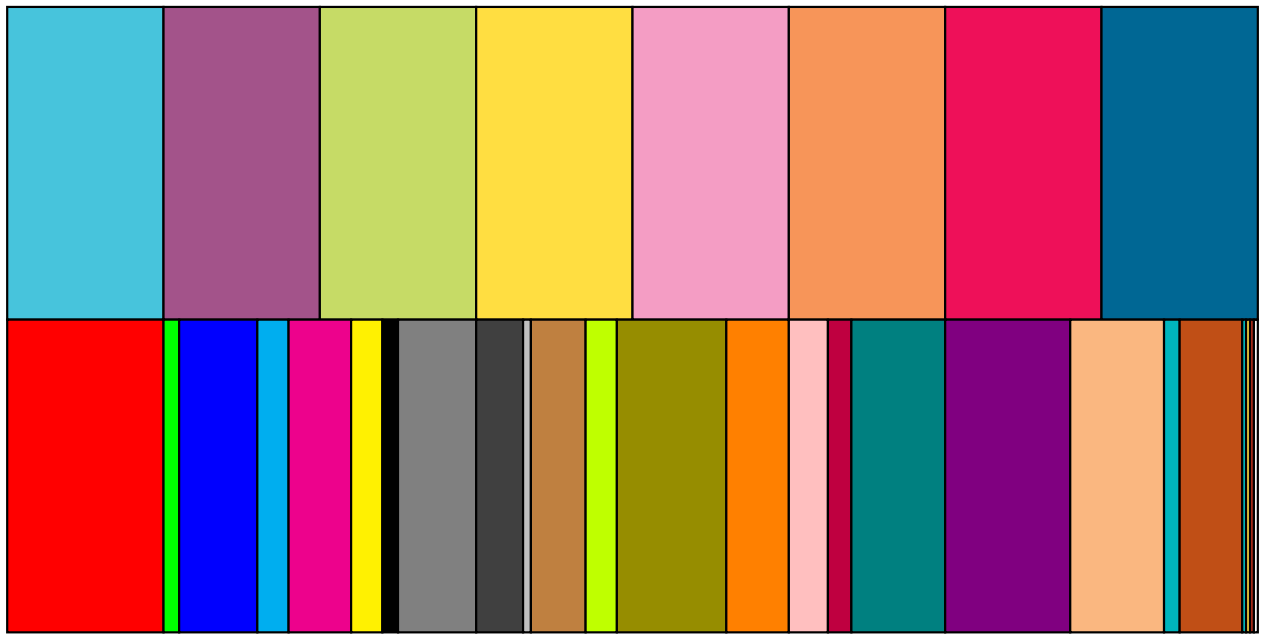

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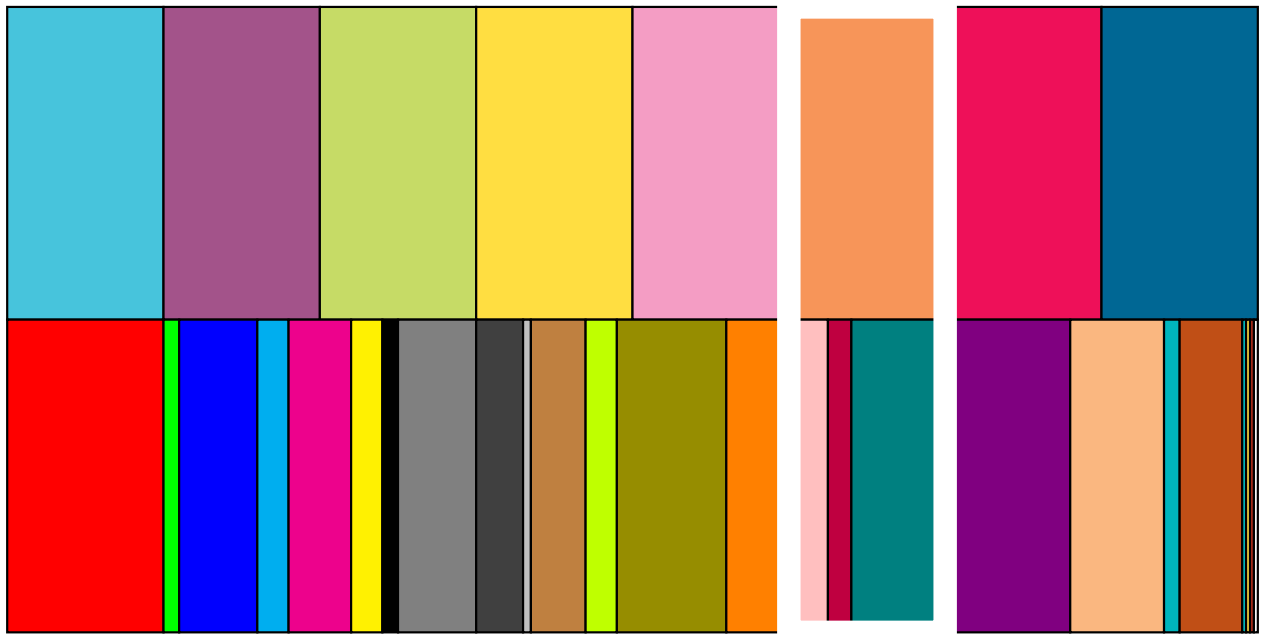

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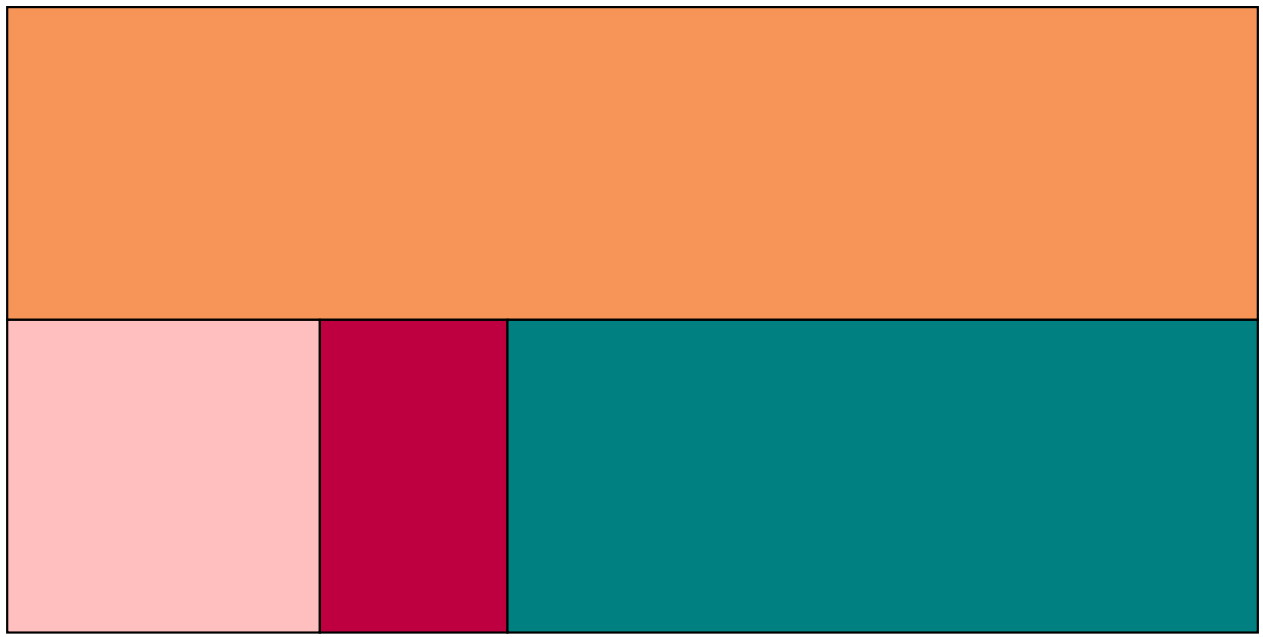
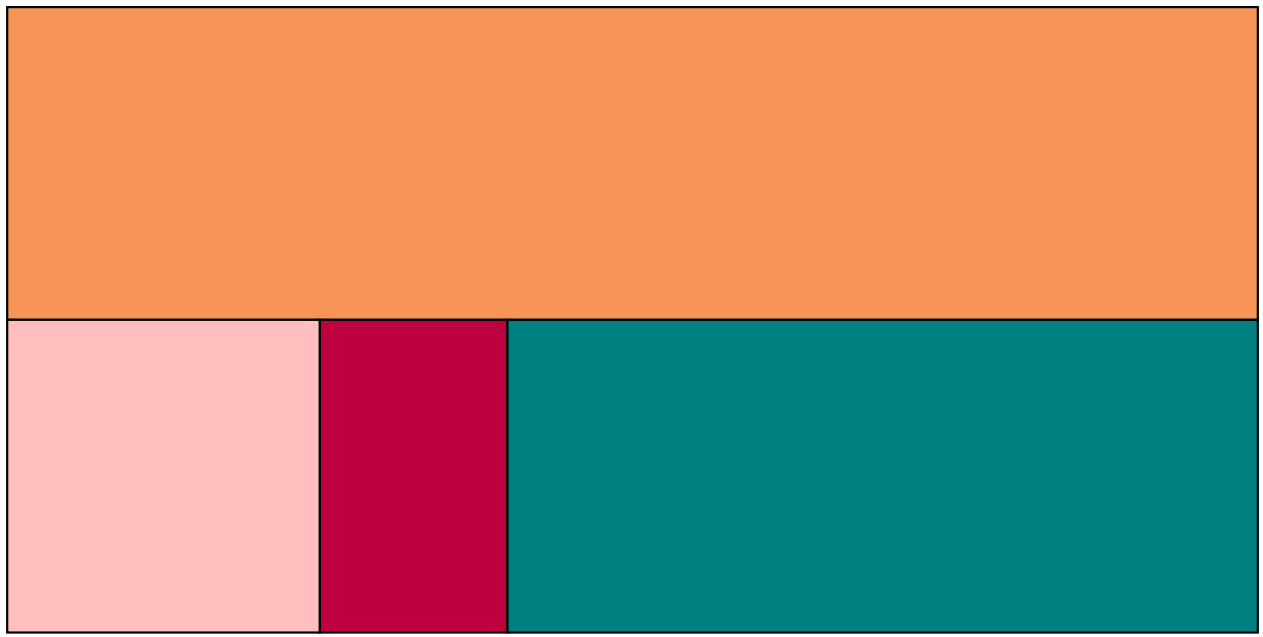


When $H(m) =$  we will set $H'(m)$ to either  ,  , or  .

Simulating H'



When $H(m) =$  we will set $H'(m)$ to either  ,  , or  . We need a randomness oracle $R : [M] \to \{0,1\}^*$ to decide which. (Querying R does not require us to query H.)

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Note that this only requires one query to H.

Converting H' collision to H

Then note that if the adversary finds $m_1, m_2 \in [M]$ such that $H'(m_1) = H'(m_2) =$  , we have that $H(m_1) = H(m_2) =$  .

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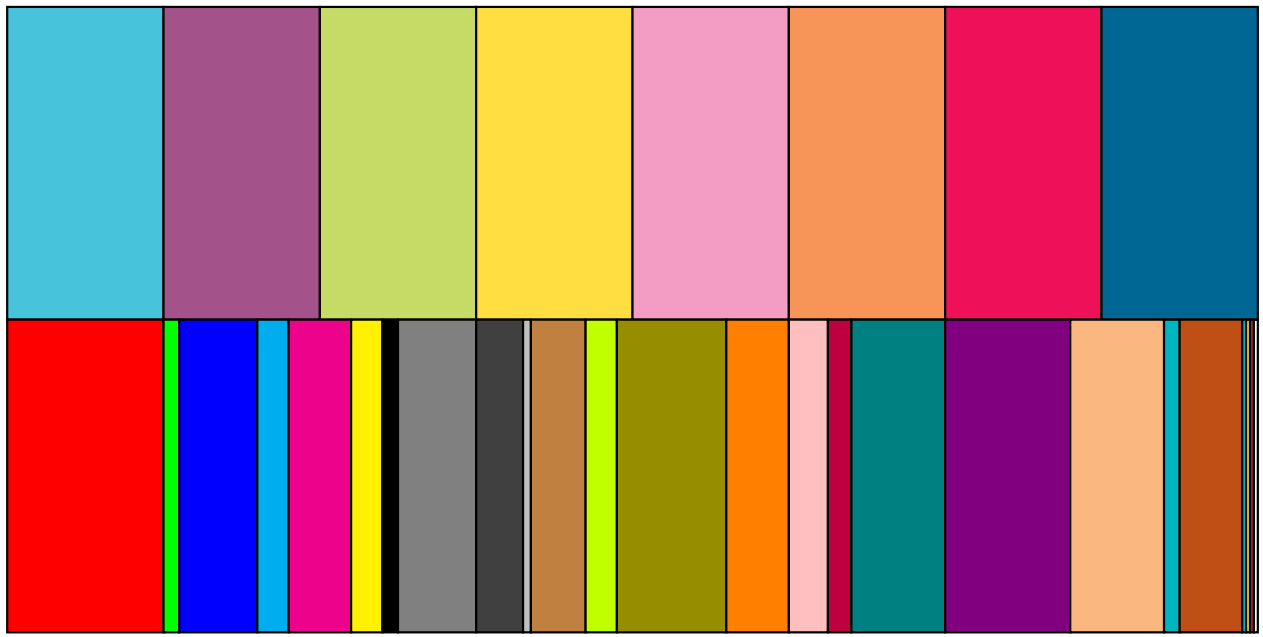
Then note that if the adversary finds $m_1, m_2 \in [M]$ such that $H'(m_1) = H'(m_2) =$  , we have that $H(m_1) = H(m_2) =$  .

But we don't always have this property.

Converting H' collision to H



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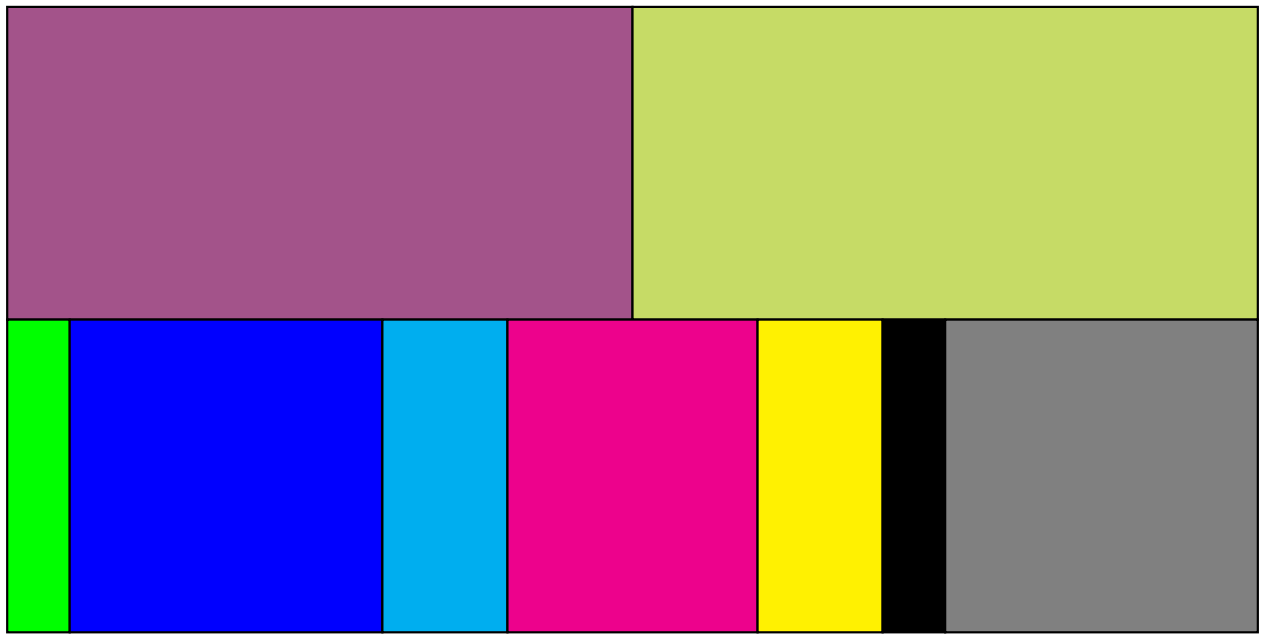

Say the adversary finds $m_1, m_2 \in [M]$ such that $H'(m_1) = H'(m_2) =$  . What does this mean for $H(m_1)$ and $H(m_2)$?

Converting H' collision to H

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Converting H' collision to H

If $H'(m_1) = H'(m_2) =$  , we could have:

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The adversary has no way to tell which — carefully accounting, we can see that

$$\Pr[H(m_1) = H(m_2)] \geq 1/2.$$

(Probability over the R oracle and the adversary's internal randomness).

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- Pass this result into a quantum circuit that breaks apart the uniform distribution to get the desired min-entropy k distribution.
- Perform garbage collection and pass result to the adversary.

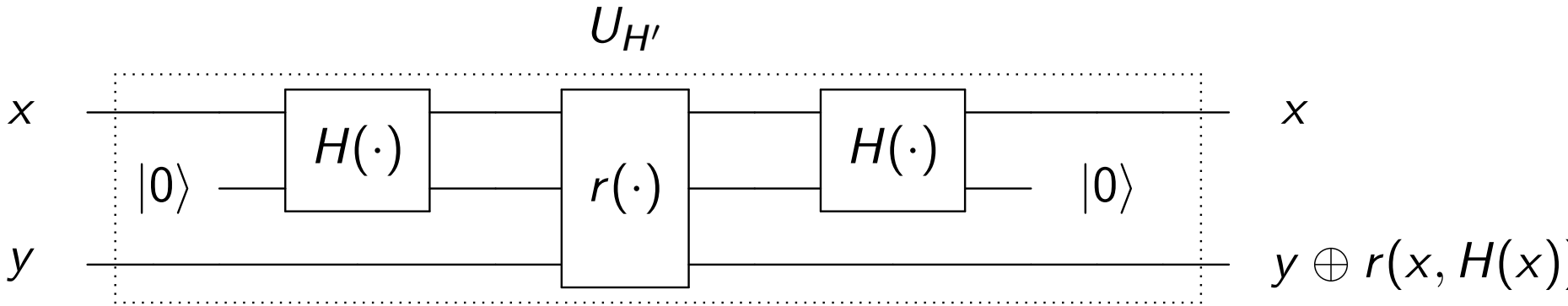

Figure: Quantum circuit that implements function $U_{H'}$ using two oracle calls to U_H.

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Since we know that $2^{k/3}$ queries are needed to break the collision resistance of H with constant probability, the adversary needs to make at least $2^{k/3}$ queries to H'.

Open Questions

- Can we get distribution conversions that are *time efficient* (not just query efficient)?
- Can we get bounds when distributions aren't entirely known?
- Lower bounds in terms of $\beta(D)$ rather than k?

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Thank you!