Attacks on the Mersenne-based AJPS cryptosystem

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April 9, 2018

Overview

May '17

Aggarwal, Joux, Prakash, Santha [AJPS17]

- Propose *potentially quantum-safe* public-key cryptosystem based on Mersenne numbers and NTRU [HPS98].

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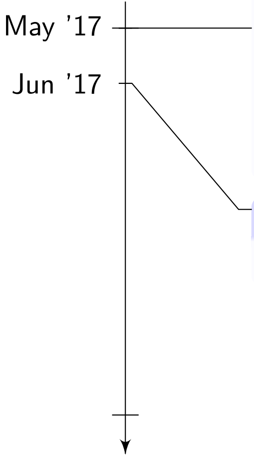

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Describe an experimental lattice-reduction attack.

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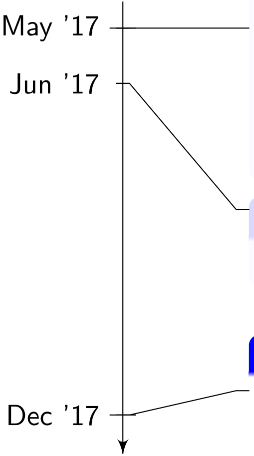

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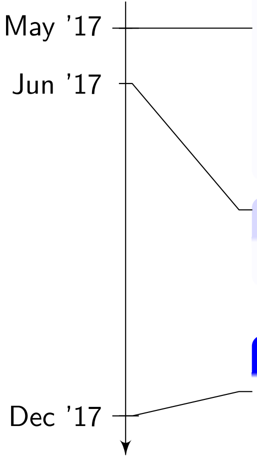

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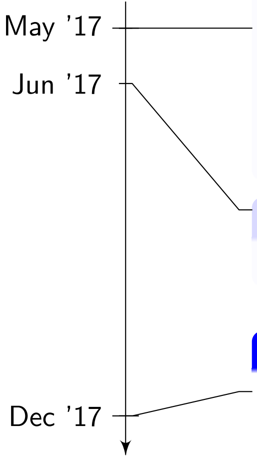

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$a \in R$	bin. rep.	$\lvert a \rvert$
0	0...000	0
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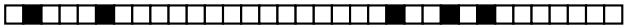
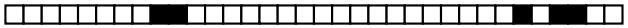
$f =$  , $g =$ 

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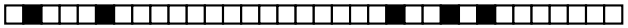
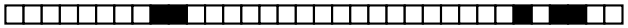
$f =$  , $\boldsymbol{g} =$ 

$\boldsymbol{h} = \frac{f}{\boldsymbol{g}} =$ 

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Brute force attack: Guess a $g \in R$ with $|g| = w$, check whether $|g\boldsymbol{h}| = w$. time: $\binom{n}{w}$.

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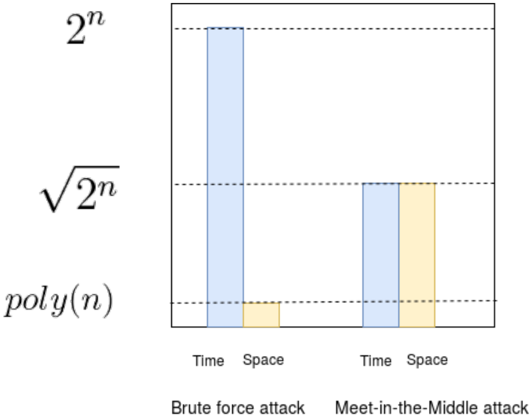
Meet-in-the-Middle attack

Improved time complexity, at the cost of greater space complexity.



MITM in the subset-sum problem

Subset-sum problem

Given $z_1, \ldots, z_n \in \mathbb{Z}$
Find $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} z_i = 0$.

$\mathbf{z_1}$	**6**
z_2	2
$\mathbf{z_3}$	**−1**
z_4	10
z_5	9
$\mathbf{z_6}$	**−5**

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8	$\{1, 2\}$
7	$\{1, 2, 3\}$

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$-\sum_{i \in \{4\}} z_i = -10$

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Output: $\{1, 3\} \cup \{6\}$

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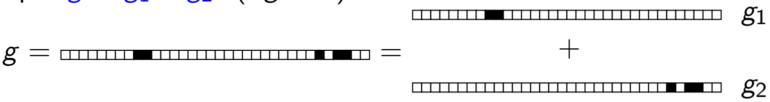

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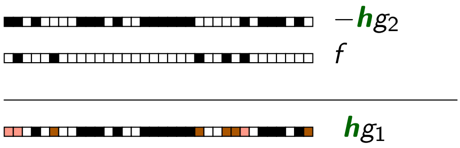

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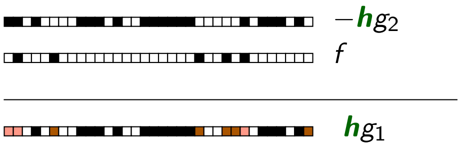

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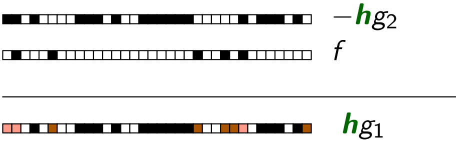

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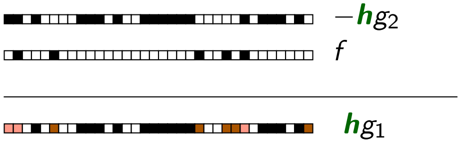

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- Then $\Delta_{Hamm}(-hg_2, hg_1) \leq 2w + c\sqrt{w}$ with error probability $\leq e^{-c/8}$.
- Informally: $-hg_2 \approx hg_1$.

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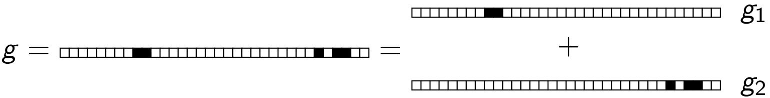
- Split a possible $g = g_1 + g_2$ in two parts, where $g_1 \in \{0,1\}^{n/2} \times 0^{n/2}$ and $g_2 \in 0^{n/2} \times \{0,1\}^{n/2}$.



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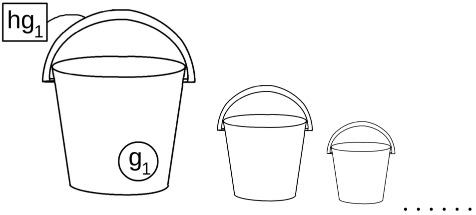

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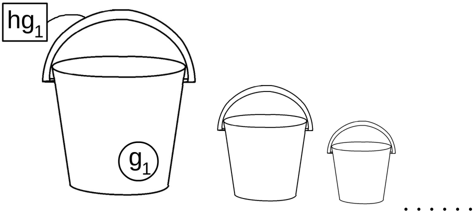

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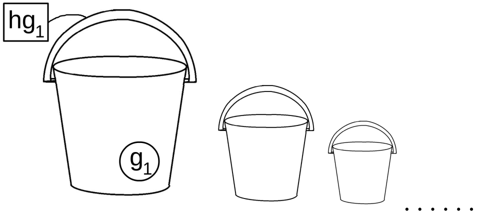

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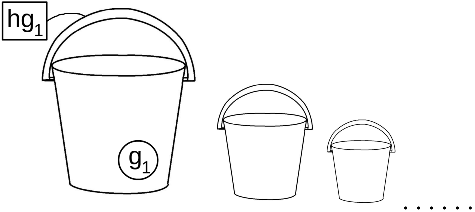

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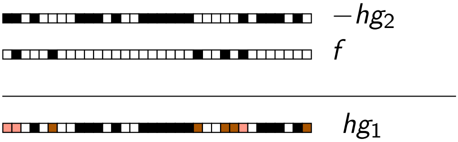
Problem: There are **many** $t \approx -hg_2$, and most of the buckets $L[t]$ are empty

Locality Sensitive Hashing

$hg_1 = -hg_2 + f$



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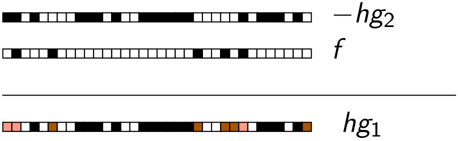

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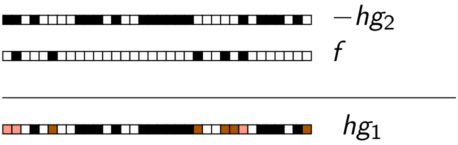

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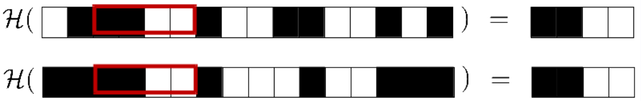

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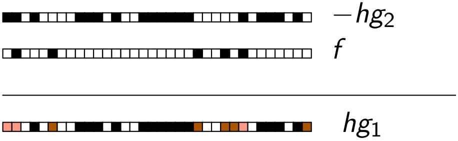

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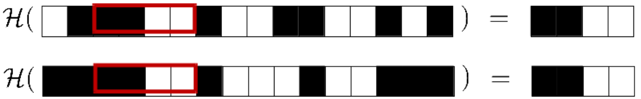

Hope: $\mathcal{H}(hg_1) = \mathcal{H}(-hg_2)$

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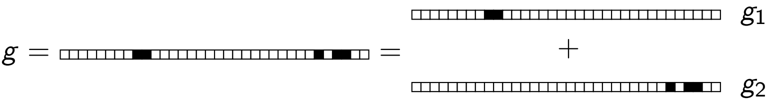

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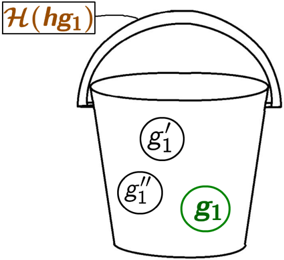

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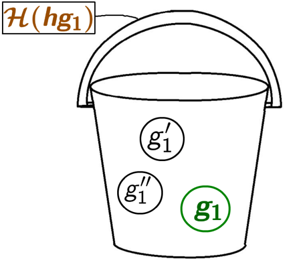

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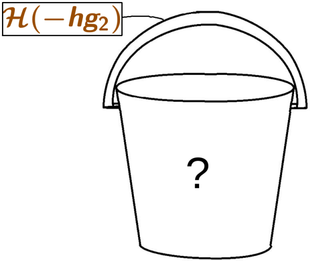

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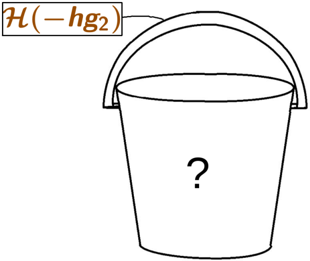

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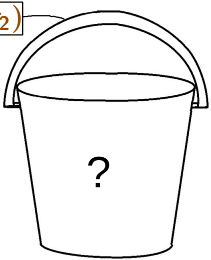

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This algorithm breaks the AJPS system in time $\binom{n/2}{w/2} \approx n^{\sqrt{n}/8}$

Overview

Attack	Authors	Running time	
		Classical	Quantum

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Lattice attack	BCGN	$2^{\sqrt{n}}$?	$2^{\sqrt{n}/2}$?
	Our analysis	$2.01^{\sqrt{n}}$	$2.01^{\sqrt{n}/2}$

Open Questions

- We analyzed the lattice attack of Beunardeau et al. over randomly chosen keys.

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- We analyzed the lattice attack of Beunardeau et al. over randomly chosen keys.
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- Aggarwal et al. improved their cryptosystem [AJPS17], allowing to encrypt more bits.
  What is the security of this improved system?

Main lesson

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Collisions don't need to be exact to apply a Meet-in-the-Middle attack

References

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