# Separating the Fan Theorem and Its Weakenings II

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**Abstract**

Varieties of the Fan Theorem have recently been developed in reverse constructive mathematics, corresponding to different continuity principles. They form a natural implicational hierarchy. Earlier work showed all of these implications to be strict. Here we re-prove one of the strictness results, using very different arguments. The technique used is a mixture of realizability, forcing in the guise of Heyting-valued models, and Kripke models.
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## 1 Introduction

The Fan Theorem states that, in $2^{<\omega}$, every bar is uniform.[1] It has been important in the foundation of constructive mathematics ever since it was first articulated (by Brouwer), and so it is no surprise that with the development of reverse mathematics in recent years it has become an important principle there. In particular, various weakenings of it have been shown to be equivalent to some principles involving continuity and compactness [2, 4, 10]. These weakenings all involve strengthening the hypothesis, by restricting which bars they apply to. The strictest version, $\mathrm{FAN}_\Delta$ or Decidable Fan, is to say that the bar $B$ in question is **decidable**: every node is either in $B$ or not. Another natural version, $\mathrm{FAN}_{\Pi^0_1}$ or $\Pi^0_1$ Fan, is to consider $\Pi^0_1$ **bars**: there is a decidable set $C \subseteq 2^{<\omega} \times \mathbb{N}$ such that $\sigma \in B$ iff, for all $n \in \mathbb{N}$, $(\sigma, n) \in C$. Nestled in between these two is $\mathrm{FAN}_c$ or $c$-Fan, which is based on the notion of a $c$**-bar**, which is

[1] A bar is a set of nodes which contains a member of every (infinite) path, and it is uniform if it contains a member of every (infinite) path by some fixed level of $2^{<\omega}$.

a particular kind of $\Pi^0_1$ bar: for some decidable set $C \subseteq 2^{<\omega}$, $\sigma \in B$ iff every extension of $\sigma$ is in $C$. It is easy to see that the implications

$$\mathrm{FAN}_{\mathrm{full}} \implies \mathrm{FAN}_{\Pi^0_1} \implies \mathrm{FAN}_c \implies \mathrm{FAN}_\Delta.$$

all hold over a weak base theory. What about the reverse implications? (We always include the implication of $\mathrm{FAN}_\Delta$ from IZF when discussing the converses of the conditionals above, IZF being our choice of base theory for non-implications.)

There had been several proofs that some of the converses did not hold [1,3,6]. These were piecemeal, in that each applied to only one converse, or even just a weak form of the converse, and used totally different techniques, so that there was no uniform view of the matter. This situation changed with [14], which provided a family of Kripke models showing the non-reversal of all the implications. It was asked there whether those models were in some sense the right, or canonical, models for this purpose; implicit was the question whether the other common modeling techniques, realizability and Heyting-valued models, could provide the same separations.

Here we do not answer those questions. We merely bring the discussion along, by providing a different kind of model. It should be pointed out early on that the only separation provided here is that $\mathrm{FAN}_\Delta$ does not imply $\mathrm{FAN}_c$, although we see no reason these arguments could not be extended to the other versions of FAN.

There are several ways that the model here differs from those of [14]. In the earlier paper, a tree with no simple paths was built over a model of classical ZFC via forcing, and the non-implications were shown by hiding that tree better or worse in various models of IZF. In particular, we showed there that $\mathrm{FAN}_\Delta$ does not imply $\mathrm{FAN}_c$ by including that tree as the complement of a $c$-bar in a gentle enough way that no new decidable bars were introduced. Here, we start with a model of $\neg\mathrm{FAN}_\Delta$, and extend it by including paths that miss decidable (former) bars. If this is done to all decidable bars, $\mathrm{FAN}_\Delta$ can be made to hold. If this is done gently enough, counter-examples to $\mathrm{FAN}_c$ will remain as counter-examples.

The other difference is in the techniques used. It is like a Kripke model built using Heyting-valued extensions of a realizability model. This is not the first time that some of these techniques have been combined (see [17] for references and discussion). This is the first time we are aware of that all three have been combined. Perhaps that in and of itself makes this work to be of some interest.

An earlier version of this work was presented at LFCS '18 [13]. The previous paper seems to be correct, except for the last paragraph. The reason I did not check it carefully enough at the time was that it seemed so obviously true. The argument was that $\mathrm{FAN}_c$ fails because there is a $c$-set which is avoided only by (the characteristic function of) the halting problem. Under recursive realizability (Kleene's $K_1$) the solution to the halting problem is not realized to be a function, so that $c$-set is a $c$-bar in that model. It seemed clear at the time that the same would hold in a generic extension of the recursive realizability model, that no generic would introduce the halting problem's solution. This turned out to be wrong though. The arbitrary covers used earlier (still defined in the next section, and used there only) allow for the importation of extra information through the realizer that something is in a chosen cover. Hence one is forced to use some kind of canonical cover, disallowing the presence of extra information. Over $K_1$, $\mathrm{FAN}_c$ fails because truth is controlled by computability; in order for $\mathrm{FAN}_c$ to continue to fail over generic extensions of $K_1$, it is inadequate to use arbitrary covers – rather, computability must again be leveraged, and so has to play a role in the semantics.

That change led to several others in its wake. Then as now, nodes in the Kripke model are based upon values from Heyting algebras. Earlier, the bottom value $\perp$ of the Heyting algebras was disallowed to be used. That means that for something to be a node, its components must be different from $\perp$. That check is not computable, not even computably enumerable, but merely co-c.e: at some point in evaluating the semantics, one would have to go out beyond a given level $n$ of a tree to a later level by which all nodes from level $n$ with only finitely many descendants are seen as such, and there is no computable way to do this (uniformly in $n$). This messes the semantics up. It turns out to be easier to allow inconsistent nodes (that is, nodes that have $\perp$ as a component). By so doing, a consistency check as sketched above can be avoided. This is especially interesting, because in a classical meta-theory including inconsistent nodes or not is irrelevant: an easy induction shows that an inconsistent node forces everything, and, using that, a separate induction shows that the consistent nodes force the same things whether or not nodes forcing everything are present. So we apparently have here a live example of a difference in semantics depending upon the logic of the meta-theory.

A parallel adjustment over the earlier version happened on the level of the Heyting algebras. To summarize the changes detailed above, the Kripke semantics was changed by including computability in order to retain the $c$-bar, and this led to the inclusion of inconsistent nodes. Similarly, in the Heyting semantics, if we include more representatives for $\perp$ then that in turn allows us to include computability, in the form of dropping the double negation in the definition of a cover, while still retaining that these models eliminate decidable bar. The root of this adjustment is the same for both they Heyting and the Kripke semantics: given a computable tree, there is in general no computable way to tell whether the tree is finite or infinite beyond a given node, a consistency check. The double negation was introduced as a way around this consistency check. If a tree beyond a node is indeed finite, you don't need to compute a witness of such, as anything would realize that there was not not such a level. If one though allows nodes that represent $\perp$ then one can use them in the semantics, and avoid the need for computation, even without the double negation. How does this play out in particular? In the proof that the generic is a path through the tree, we have to show that for every level $k$ the generic contains a unique node from that level. If we had to use only non-$\perp$ values, we would be obligated to be able to distinguish between those nodes and the ones representing $\perp$. If we allow $\perp$-nodes, then it is easy to show that every node on level $k$ forces the generic to contain a unique node from level $k$, namely itself.

Although I was led to inconsistent nodes and representatives for $\perp$ by efforts to correct oversights and make sure the proof is correct, they slowly took on a life of their own, bringing along other benefits. The first I noticed was that they simplified the technicalities, since one needs separate, additional clauses to exclude them, and that complexity unsurprisingly propagates. After the fact, they brought along also aesthetic and philosophical benefits. Constructivism got its start through Brouwer's philosophy, which founded mathematics upon our intuitions of time and space, elements that Brouwer got from Kant. A central concept for Kant was the thing-in-itself, *das Ding an sich*, which is the source of our sensory perceptions, yet cannot be accessed directly. This unknowable other seems to be ever-present, while always remaining unapproachable. In a sense, inconsistent nodes are a reflection of this idea. They have no structure that we can discern: they all force all formulas to be true, hence they are internally indistinguishable from each other. They are unavoidable: every node can be extended to become inconsistent. They seem in this model to be a sort of black hole: once you get in, there is no getting out, and all of the normal rules of logic no longer apply. More broadly, they can be taken as a pale reflection within the limited reach of one Kripke model of all of the realms of being we might never experience. It's like going through a black hole and out the other side, like going backwards in time to before the Big Bang, like flipping into the mirror realm of dark matter and energy, like stepping outside of your own consciousness and its limiting isolation. Of course, don't try to grab onto this feeling too tightly. Using classical logic, it makes no difference if these nodes are there or not, so if you impose that stricter regimen on them, they will evaporate like ghosts. We study everything else, all the consistent nodes in the Heyting and Kripke models, for it is there we find structure and meaning; then at the end, at inconsistent nodes, it all collapses into one.

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## 2 Kripke structures of constructive models

While the general theory of models of constructive systems is often itself presented constructively (for example in [7, 9, 18]), particular models are often built within a classical meta-theory, because essential use is made of classical constructions (as in [5] or [14], for instance). For Kripke models, that means, working classically, giving a partial order, and associating to each node a classical model (for the similarity type in question), along with a family of transition functions; this then determines a model of constructive logic. Our current setting is different.

A warning shot is given by the fact that the root of this model is the recursive realizability model, that is, the one based on Kleene's PCA $K_1$, which we therefore call $\mathcal{M}_{K_1}$.[2] To the degree we work within this model, we must work constructively, and not classically. This point could be finessed, though, by insisting that $\mathcal{M}_{K_1}$ was itself built within a classical theory.

More crucially, the structures at the nodes will be determined by Heyting-valued extensions of $\mathcal{M}_{K_1}$. The intuition behind the construction is that the structure at a node is simply a Heyting-valued model; if the node $\tau$ extends $\sigma$, then the structure at $\tau$ is supposed to be a Heyting-valued extension of that at $\sigma$. There is an issue though with formalizing matters just that way: such structures are no longer of the right type. A structure of the language of set theory would, among other things, determine, for objects $a$ and $b$, whether $a \in b$. A Heyting-valued model, though, determines whether $\mathcal{H} \Vdash_H$ "$a \in b$", where $\mathcal{H}$ is a value in the Heyting algebra $\mathcal{T}$ in question, and $\Vdash_H$ is the standard semantics for Heyting-valued models.

**Notation:** Often a Heyting-valued model is understood as providing a value $[\![\phi]\!]$ in a fixed Heyting algebra for a sentence $\phi$. The notation $\mathcal{H} \Vdash_H \phi$ is just another way to say $\mathcal{H} \leq [\![\phi]\!]$, in words that $\mathcal{H}$ is *stronger* than or *extends* $[\![\phi]\!]$, and comes from classical forcing. The standard exposition of classical forcing involves a partial order, and $p \Vdash \phi$, where $p$ is in the partial order, is defined inductively on $\phi$. A less-common albeit equivalent alternative is Boolean-valued models, in which a Boolean value $[\![\phi]\!]$ is defined, the Boolean algebra being a Boolean completion of the partial order. The relationship here is again $p \Vdash \phi$ iff $p \leq [\![\phi]\!]$. Heyting-valued models are really nothing other than classical forcing, with Excluded Middle taken away from the model so built.

It would be interesting to extend the notion of Kripke models to allow not just classical structures at each node but also structures for constructive theories, such as Heyting-valued models; presumably, we believe, this can be done, and we leave this for future work. For the sake of efficiency at the moment, we address these matters by allowing the Heyting algebra to be spread within the partial order, thereby mixing the Heyting and Kripke ideas together. That is, a structure at a node will contain, in addition to a Heyting-valued model, also a Heyting value $\mathcal{H}$ from the Heyting algebra. A node can be extended by strengthening or extending the Heyting value. Truth at a node will be determined in part with reference to coverings coming from the Heyting algebra; that is, the covering or join operation from the Heyting algebra will induce a covering relation among the nodes, which will be used in the semantics. So the partial order will be doing double duty, containing both the Heyting algebra structures and their associated notions of covering, and the Kripke model part. This should not be confused with say Beth models, which have a single notion of covering applying to the entire order. Here we have different (albeit parallel) coverings, independent of the iteration which is purely a Kripke structure.

Before giving the formal definitions, we specify the idea a bit more. We

[2]Kleene defined realizability only for arithmetic, the extension to full set theory coming only later [15] (see also [16]); a description of both $K_1$ and of realizability models in general can be found in many introductions to realizability, such as [1] and [17].

will need to iterate taking Heyting-valued extensions. By way of notation, $\mathcal{T}$ will be taken to be a typical complete Heyting algebra, to be consistent with the notation below. Some of these Heyting algebras $\mathcal{T}$ will show up only in some previously constructed Heyting-valued extension. So we assume we have a definable collection of Heyting algebras, say with definition $\phi(\mathcal{T})$, which IZF proves to be a set.[3] Each node $p$ will be a string $\langle(\mathcal{T}_0, \mathcal{O}_0), (\mathcal{T}_1, \mathcal{O}_1), ..., (\mathcal{T}_n, \mathcal{O}_n)\rangle$ such that each Heyting value $\mathcal{O}_i$ forces that the next $\mathcal{T}_{i+1}$ is an allowable Heyting algebra, as given by $\phi$. A child of $p$ is determined by, optionally, extending some of the $\mathcal{O}_i$'s, and, optionally, including another pair $(\mathcal{T}_{n+1}, \mathcal{O}_{n+1})$ onto the string.

More formally, let $\phi(x)$ be a formula such that IZF proves "if $\phi(x)$ then $x$ is a complete Heyting algebra, and $\phi$ is satisfied by only set-many objects." We will have occasion to consider $\phi$ as evaluated in a Heyting-valued extension, and so as applied to a term. Even if there are only set-many objects satisfying $\phi$ in this extension, there could still be class-many such terms. In order to keep this construction fully set-sized, we will allow only set-many terms $\mathcal{T}$ to be applied to $\phi$. In more detail, suppose we assert in some context that $\phi(\mathcal{T})$. If this context is some ground model $\mathcal{M}$ of IZF, then there are only set-many such $\mathcal{T}$'s within this model. Else the context will be some Heyting-valued extension, given by a Heyting value $\mathcal{H}$ – $\mathcal{H} \Vdash_H \phi(\mathcal{T})$ – within some other context, a ground model $\mathcal{M}$ of IZF. Then $\mathcal{M}$ will satisfy that there are only set-many terms forced by any $\mathcal{H}$ satisfying $\phi$.

**Remark 1.** *In the end, our ultimate model starts from $\mathcal{M}_{K_1}$, itself built within a classical meta-universe $V$. The construction to follow will be carried out in later sections within $\mathcal{M}_{K_1}$. At the same time, the construction of this section can be done within any model $\mathcal{M}$ of IZF in place of $\mathcal{M}_{K_1}$. So we will give the definition in terms of an unspecified $\mathcal{M}$. The reader may of course choose to think of $\mathcal{M}$ as $\mathcal{M}_{K_1}$, the only case we will actually use.*

**Definition 1.** *Definition of the nodes, and their associated models, by induction on $\omega$:*

*The unique node of length 0 is the empty sequence $\langle\rangle$, with associated model $\mathcal{M}_{\langle\rangle} = \mathcal{M}$.*

*Inductively, given the set of nodes of length $n$, a node $p$ of length $n+1$ will be a string of the form $\langle(\mathcal{T}_0, \mathcal{O}_0), (\mathcal{T}_1, \mathcal{O}_1), ..., (\mathcal{T}_n, \mathcal{O}_n)\rangle$ such that $p \upharpoonright n$ is a node, and, in $\mathcal{M}$, $\mathcal{O}_0 \Vdash_H$ "$\mathcal{O}_1 \Vdash_H$ ..."$\mathcal{O}_{n-1} \Vdash_H$ "$\phi(\mathcal{T}_n)$ and $\mathcal{O}_n$ is a Heyting value in the Heyting algebra $\mathcal{T}_n$""...". The model $\mathcal{M}_p$ associated to $p$ is the forcing extension by $\mathcal{T}_n$, with truth determined by $\mathcal{O}_n$, as evaluated within the model for $p \upharpoonright n$.*

*If $p \upharpoonright n \Vdash \mathcal{O}_n = \bot$ then $p$ and all of its descendants are* **inconsistent***.*

We abbreviate the iterated forcing $\mathcal{O}_0 \Vdash_H$ "$\mathcal{O}_1 \Vdash_H$ ..."$\mathcal{O}_{n-1} \Vdash_H \psi$"..." as $\langle\mathcal{O}_0, ..., \mathcal{O}_{n-1}\rangle \Vdash_H \psi$. The reason for the subscript $H$ is to emphasize that this

[3] We do not need to assume that $\phi$ defines an inhabited set. In the degenerate case of $\phi$ picking out nothing, then the definitions given collapse to a one-node Kripke model, with no iteration. In the indeterminate case when we don't know whether $\phi$ yields the empty set, then we just don't know whether the Kripke model has one node or more.

notion of truth is given by iterated forcing. In contrast, for example, truth in $\mathcal{M}_{K_1}$ is given by realizers, and will be written as $e \Vdash_r \psi$. One important instance of the latter will be iterated forcing over $\mathcal{M}_{K_1}$. So truth in the model given by forcing with $\mathcal{T}$ over $\mathcal{M} = \mathcal{M}_{K_1}$ would be written as $e \Vdash_r$ "$\mathcal{O} \Vdash_H \psi$". Another example will be the upcoming Kripke model. The partial order for the Kripke model will be built on the same set of nodes just defined. The difference to the Heyting-valued models above is that the associated Heyting-valued models $\mathcal{M}_q$ for all $q \geq p$ will be combined to form a Kripke model $\mathcal{M}^p$ (note the subtle change in name: superscript instead of subscript) at $p$. Truth in the Kripke model will look like $p \Vdash_K \phi$. When this is evaluated within $\mathcal{M}_{K_1}$, it can look like $e \Vdash_r$ "$p \Vdash_K \phi$". As long as we remain agnostic about the nature of $\mathcal{M}$, we will use the modeling notation $\models_{\mathcal{M}}$. So iterated forcing will look like $\mathcal{M} \models_{\mathcal{M}}$ "$\mathcal{O} \Vdash_H \psi$", and truth in the Kripke model will be $\mathcal{M} \models_{\mathcal{M}}$ "$p \Vdash_K \phi$"

By our various conventions, there are only set-many nodes.

By way of notation, we will consider $p$ as being $\langle(\mathcal{T}_0^p, \mathcal{O}_0^p), (\mathcal{T}_1^p, \mathcal{O}_1^p), ..., (\mathcal{T}_n^p, \mathcal{O}_n^p)\rangle$. Typically mention of the $\mathcal{T}_i^p$'s will be suppressed, as they are implicit in the choice of the $\mathcal{O}_i^p$'s, so that $p$ of length $n$ will be $\langle \mathcal{O}_0^p, ..., \mathcal{O}_{n-1}^p\rangle$.

**Definition 2.** *The (Kripke) partial order on the set of nodes: For $q$ to be an extension of $p$, written $q \geq p$, $q$ has to be at least as long as $p$, and, for $i$ less than the length of $p$, $q \restriction i \Vdash_H \mathcal{O}_i^q \leq \mathcal{O}_i^p$. (For this to make sense, implicitly $q \restriction i \Vdash_H \mathcal{T}_i^q = \mathcal{T}_i^p$.)*

We leave it to the reader to show that this is indeed a partial order. Notice that an extension of a node is indicated with the standard notation for partial orders, $\geq$, in contrast with the strengthening of a Heyting value, which is indicated with the standard notation for forcing, $\leq$.

This p.o. is in $\mathcal{M}$. Since a model embeds into any Heyting-valued extension, the p.o. is also in any of the models associated with a node. Furthermore, consider the p.o. restricted to a node (i.e. the extensions of any node, including itself). This restriction is definable in the node's model, uniformly from the node. That is, given any node as a parameter, the node's model can figure out the rest of the p.o.

We are finally in a position to define the model. This will be done within $\mathcal{M}$ inductively on the ordinals $\alpha$. (An ordinal is taken to be a transitive set of transitive sets.) We define the members $\mathcal{M}_\alpha^p$ of the model at node $p$ of rank $\alpha$ (where we associate $\alpha$ with its canonical image in each of the associated models), along with the transition functions $f_{pq}$ from $\mathcal{M}_\alpha^p$ to $\mathcal{M}_\alpha^q$. We will usually drop the subscripts and just write $f$ as a polymorphic transition function. Similarly, we will not adorn $f$ with any $\alpha$, since the definition of $f$ will be uniform in $\alpha$. Do not confuse the associated models $\mathcal{M}_p$ from above with the $\mathcal{M}^p$ about to be defined.

**Definition 3.** *The universe $\mathcal{M}^p$ of the model at node $p$: First we define $\mathcal{M}_\alpha^p$ inductively on ordinals $\alpha$. A member $\sigma$ of $\mathcal{M}_\alpha^p$ is a function with domain the p.o. restricted to $p$ (i.e. $p$ and its extensions). Furthermore, $\sigma(q) \subseteq \bigcup_{\beta<\alpha} \mathcal{M}_\beta^q$. In order to fulfill the basic Kripke condition, if $\tau \in \sigma(q)$, and $r \geq q$, then $f(\tau) \in \sigma(r)$. If $q \geq p$, then $f(\sigma)$ is defined to be $\sigma \restriction \mathcal{P}^{\geq q}$. Let $\mathcal{M}^p$ be $\bigcup_\alpha \mathcal{M}_\alpha^p$.*

(If you're wondering whether there are any such members, or whether instead the definition is vacuous, see the proof of IZF below; the reader is invited to think through now why the empty set is a set within this formalism.)

**Proposition 2.** *$\mathcal{M}^p$ is the image of $\mathcal{M}^{\langle\rangle}$ under the transition function.*

*Proof.* Given $\sigma \in \mathcal{M}^p$, let $\sigma^+ \in \mathcal{M}^{\langle\rangle}$ be such that $\sigma^+(q) = \sigma(q)$ when $q \geq p$, else $\sigma^+(q) = \emptyset$. The (possibly non-constructive) case split can be avoided by letting $\sigma^+(q)$ be $\{x \mid x \in \sigma(q) \wedge q \geq p\}$. □

Because this model has aspects of both a Kripke and a Heyting-valued model, it is in actuality neither. So to give the semantics, we cannot rely on any standard definition already extant in the literature. Rather, we have to give an independent, inductive definition of satisfaction.

This is the point at which this section branches off from the model to be constructed later on. In giving the semantics later for the ultimate model of interest, we will make use of particular aspects of the particular class of Heyting algebras chosen. It is not clear to the author whether this could all be subsumed under some broader definition, applicable to all classes of Heyting algebras, or at least generalized by axiomatizing the features of the Heyting algebras we will actually need. But even if this is possible, it is another matter whether it is desirable. Sometimes it's easier to understand and not make mistakes with a more concrete argument, an individual case, than a general case, especially the first time through it. For these reasons, we defer the semantics until after the development of the particular Heyting algebras we will use. (See the beginning of the section on the Kripke semantics for further detail.)

At the same time, there is a closely related semantics that is general, applicable to all choices of Heyting algebras. It seems like a good idea to present it here, in part for the record, in part for possible future use, and in part because it might well be the same thing as the semantics we actually use in this paper (again, see the Kripke semantics section for more discussion). Even if it's not, it is a simpler version of what comes later, so the reader might choose to get used to this mix of ideas by reading it through. This will occupy the remainder of this section and is strictly speaking not necessary for the rest of the paper.

We will need the notion of a node being covered by a set of nodes, akin to an open set in a topological space being covered by a collection of open sets, or, more generally, a member of a Heyting algebra being (less than) the join of a subset of the algebra.

**Definition 4.** *We define $p$ of length $n$ being covered by $P = \{p_j \mid j \in J\}$ by induction on $n$:*

- *For $n = 0, \langle\rangle$ is covered by only $\{\langle\rangle\}$.*
- *For $n = 1, p$ of the form $\langle(\mathcal{T}, \mathcal{O})\rangle$ is covered by $P$ if each $p_j$ also has length 1, and $p_j \geq p$ (so $p_j$ is of the form $\langle(\mathcal{T}, \mathcal{O}_j)\rangle$, the point being that $\mathcal{T}$ is the same), and $\{\mathcal{O}_j \mid j \in J\}$ covers $\mathcal{O}$ in the sense of $\mathcal{T}$: $\mathcal{O} \leq \bigvee\{\mathcal{O}_j \mid j \in J\}$.*

- *For a length $n+1>1$, some conditions are immediate analogues: each $p_j$ extends $p$ in the Kripke partial order, and each $p_j$ has length $n+1$. Furthermore, letting $P \restriction n$ be $\{p_j \restriction n \mid j \in J\}$, we have that $P \restriction n$ is to cover $p \restriction n$. Finally, we want to view $P$ as a term for a set in the model associated with $p \restriction n$. Recall that a term for a Heyting-valued model is an arbitrary collection of pairs $\langle \mathcal{O}, \sigma \rangle$, where $\mathcal{O}$ is a member of the Heyting algebra and $\sigma$ is (inductively) a term. If we are considering a two-step iteration, then $\sigma$ is (a term for) a pair $\langle \hat{\mathcal{O}}, \tau \rangle$, where $\hat{\mathcal{O}}$ is a value from the second Heyting algebra. This can be abbreviated by $\langle (\mathcal{O}, \hat{\mathcal{O}}), \tau \rangle$. Whereas each $p_j$ is of the form $\langle \mathcal{O}_0, \ldots, \mathcal{O}_n \rangle$, it induces a set $alt-p_j := \langle (\mathcal{O}_0, \ldots, \mathcal{O}_{n-1}), \mathcal{O}_n \rangle$, which is a term, in the language for an $n$-fold forcing iteration, with value (forced to be) an open set in $\mathcal{T}_n$. Of course, the $n$-fold iteration in question is just the model associated with $p \restriction n$. So, letting $P_n$ be $\{alt-p_j \mid j \in J\}$, $p \restriction n \Vdash_H P_n$ is a collection of open sets of $\mathcal{T}_n$. Our final condition is that $p \restriction n \Vdash_H P_n$ covers $p(n)$.*

**Definition 5.** *Implicitly in what follows, when we write "$p \Vdash_K \phi$", $K$ for Kripke, the parameters in $\phi$ are all in $\mathcal{M}^p$. Also implicit is the application of the transition function $f$, as need be.*

- *$p \Vdash_K \sigma \in \tau$ iff $p$ is covered by some $Q$, and for all $q \in Q$ there is a $\sigma_q \in \tau(q)$ such that $q \Vdash_K \sigma = \sigma_q$.*
- *$p \Vdash_K \sigma = \tau$ iff for all $q \geq p$ and all $\upsilon \in \sigma(q)$, $q \Vdash_K \upsilon \in \tau$, and vice versa.*
- *$p \Vdash_K \phi \wedge \psi$ iff $p \Vdash_K \phi$ and $p \Vdash_K \psi$.*
- *$p \Vdash_K \phi \vee \psi$ iff $p$ is covered by some $Q$, and for each $q \in Q$ either $q \Vdash_K \phi$ or $q \Vdash_K \psi$.*
- *$p \Vdash_K \phi \rightarrow \psi$ iff for all $q \geq p$ if $q \Vdash_K \phi$ then $q \Vdash_K \psi$.*
- *$p \Vdash_K \forall x\ \phi(x)$ iff for all $q \geq p$ and $\sigma \in \mathcal{M}^q$ $q \Vdash_K \phi(\sigma)$.*
- *$p \Vdash_K \exists x\ \phi(x)$ iff $p$ is covered by some $Q$ and for all $q \in Q$ there is some $\sigma$ such that $q \Vdash_K \phi(\sigma)$.*

**Lemma 3.** *Each node satisfies the equality axioms.*

**Lemma 4.** *If $Q$ covers $p$, and for each $q \in Q$ we have $q \Vdash_K \phi$, then $p \Vdash_K \phi$.*

*Proof.* By a straightforward induction on $\phi$. □

**Corollary 5.** *Each node satisfies constructive logic.*

*Proof.* Straightforward. □

**Theorem 6.** *This structure models IZF.*

*Proof.* For many of the axioms, we will merely give the construction of the witness. We leave it to the reader to check that the witness given satisfies the conditions listed above to be a set in the universe, and that the witness satisfies the formula it is meant to be a witness for.

Empty Set: In $\mathcal{M}$, the constant function with domain the entire tree always returning the empty set is an object of rank 0, and represents the empty set.

Infinity: First, we show that each $n \in \omega$ is canonically represented by a set in the model, inductively on $n$. The case $n = 0$ is done by the empty set, above. Suppose for $i \leq n$, $i$ is represented in the model at the root by $\sigma_i$ of rank $i$. Then $n+1$ will be represented by $\sigma_{n+1}$, where $\sigma_{n+1}(p)$ will be $\{f(\sigma_0), f(\sigma_1), ..., f(\sigma_n)\}$ (where once again $f$ is the (polymorphic) transition function). Finally, $\omega$ will be represented by $\sigma_\omega$, where $\sigma_\omega(p) = \{f(\sigma_n) \mid n \in \omega\}$.

Pair: If $\sigma, \tau \in \mathcal{M}^p$, then the pair is given by $\rho$, where $\rho(q) = \{f(\sigma), f(\tau)\}$.

Union: For $\sigma \in \mathcal{M}^p, \bigcup \sigma$ is given by $(\bigcup \sigma)(q) = \bigcup \{\tau(q) \mid \tau \in \sigma(q)\}$.

Extensionality: This follows fairly directly from the definition of forcing equality.

$\in$-Induction: The sets in the model are built in $\mathcal{M}$ by an induction on the ordinals in $\mathcal{M}$. So induction in $\mathcal{M}$ can be used to show induction in $\mathcal{M}^{\langle\rangle}$. For a more detailed argument, the proof later of IZF for $\Vdash$ contains a more detailed exposition of $\in$-Induction, which applies straightforwardly to the proof here as well.

Power Set: Let $\sigma \in \mathcal{M}^p$. Then let $(\wp(\sigma))(q)$ be $\{\tau \in \mathcal{M}^q \mid \forall r \geq q \ \tau(r) \subseteq \sigma(r)\}$.

Separation: Given $\sigma \in \mathcal{M}^p$, and $\phi(x)$ with parameters from $\mathcal{M}^p$, let $(\text{Sep}(\sigma, \phi))(q)$ be $\{\rho \in \sigma(q) \mid q \Vdash \phi(\rho)\}$.

Collection: Suppose $p \Vdash \forall x \in \sigma \ \exists \tau \ \phi(x, \tau)$. Working in $\mathcal{M}$, for each $q \geq p$ and $\rho \in \sigma(q)$, there is an $\alpha$ such that some $\tau$ in $\mathcal{M}^q_\alpha$ satisfies $q \Vdash \phi(\rho, \tau)$. By Collection in $\mathcal{M}$, there is a bounding set for all of the $\alpha$'s needed. This bounding set can be restricted to contain only ordinals, and then expanded to be an ordinal itself, say $\beta$. Let $\Upsilon$ be such that, for $q \geq p, \Upsilon(q) = \mathcal{M}^q_\beta$, which suffices for a bounding set.

□

**Remark 7.** *As usual, $\perp$ as a symbol in the language (as opposed to a value in a Heyting algebra) can be taken to be 0=1, now that we know what 0 and 1 are. Unraveling the semantics, $\perp$ comes down to $\emptyset \in \emptyset$. In order for $p$ to force that, $p$ would have to be covered by some $Q$ such that for all $q \in Q$ there is a $\sigma_q \in \emptyset$ such that $q \Vdash_K \emptyset = \sigma_q$. Of course, there is no such $\sigma_q$. So this property holds exactly when $Q$ is itself empty. In other words, $p \Vdash_K \perp$ iff $p$ is inconsistent.*

The reason this semantics does not work for current purposes is that, under it, the last theorem of the paper does not seem to be valid. Information can be smuggled in via the covers. For something to be validated in the given semantics, it needs an arbitrary cover; within $\mathcal{M}_{K_1}$, a realizer that $q$ is in some cover $Q$ could contain computational information that might be able to do something we don't want to be able to do.

# 3 The formal topologies and their Heyting algebras

We are still working simply under IZF.

Our primary task is now to define the right $\phi$, the class of Heyting algebras we will use to build the nodes. They will be induced by the possible counterexamples $B$ to $\mathrm{FAN}_\Delta$: $B$ is a decidable set of binary strings, but is not uniform. It is safe to assume that $B$ is closed upwards. Mostly we're interested in when $B$ is in addition a bar, there famously being such a creature in Kleene's recursive realizability model. The reason that we do not include being a bar in this definition is that would then be another condition to check before being able to use $B$. This is more than just a matter of convenience, or saving a little work. When we're working within a Heyting-valued extension, given by say $\mathcal{T}$, of a realizability model, different conditions within $\mathcal{T}$ might decide whether $B$ is a bar differently, and if $B$ had to be a bar then we'd need to find an infinite path through those conditions along which $B$ became a bar, meaning either there is such a path, or we'd have to find a non-uniform bar forcing such a path, and all of a sudden the thicket starts to look impenetrable. Although it seems unaesthetic to force paths that we really don't need, this is a small price to pay for having a theorem with a proof.

Let $T$ be the complement of $B$. So $T$ is a decidable, infinite tree. We will generically shoot a branch through $T$.

We will define a formal topology $S$ from $T$. To help make this paper self-contained, we present a definition of a formal topology. Such definitions are not uniform in the literature. Here we will use the one from [9], sec. 2.1. The same reference also describes how a formal topology induces a Heyting algebra.

**Definition 6.** *Formal topology: A formal topology is a poset $(S, \leq)$ and a relation $\lhd$ between elements and subsets of $S$. (One should think of the elements of $S$ as open sets, with $\leq$ as containment and $\lhd$ as covering.) The axioms are:*

- *if $a \in p$ then $a \lhd p$,*
- *if $a \leq b$ and $b \lhd p$ then $a \lhd p$,*
- *if $a \lhd p$ and $\forall x \in p\ x \lhd q$ then $a \lhd q$, and*
- *if $a \lhd p$ and $a \lhd q$ then $a \lhd \downarrow p \cap \downarrow q$,*

*where $\downarrow p$ is the downward closure of $p$.*

**Definition 7.** *The formal topology induced by $B$:*

*Let $B$ be a decidable, upwards-closed, non-uniform set of binary strings, and $T$ its complement in $2^{<\omega}$. A* **basic member $\mathcal{O}_\mu$ of** *$S$ is given by a node $\mu \in 2^{<\omega}$, and is the set of all nodes in $T$ compatible with $\mu$, that is, all initial segments and extensions which are in $T$. Note that $\mathcal{O}_\mu$ might well be finite, which will ultimately make $\mathcal{O}_\mu$ equal to the bottom element $\perp$ of the Heyting algebra. What is often called the length of $\mu$ will throughout this paper be called* *the* **height of** $\mu$**,** $ht(\mu)$**,** *thinking of it as* $\mu$*'s height in the binary tree, and to distinguish it verbally from the length of a node* $p$ *in the Kripke partial order from the previous section.*

*The* **height of** $\mathcal{O}_\mu$ *is the height of the shortest* $\nu$ *such that* $\mathcal{O}_\nu = \mathcal{O}_\mu$*. To see why this might be different from the height of* $\mu$ *itself, if* $\mu = \nu^\frown i$ *and* $\mu$*'s sibling* $\nu^\frown(1-i)$ *is not in* $T$*, then* $\mathcal{O}_\mu = \mathcal{O}_\nu$*.*

*A member* $\mathcal{O}$ *of* $S$ *is a union of finitely many basic members of* $S$*. A witness that* $\mathcal{O} \in S$*, that is, a finite set* $\Sigma$ *such that* $\mathcal{O} = \bigcup_{\mu\in\Sigma} \mathcal{O}_\mu$*, is called a* **base** *for* $\mathcal{O}$*; note that bases are not unique. The* **height of** $\mathcal{O}$ *is the smallest natural number* $n$ *at least as big as the height of each* $\mu$ *in some base* $\Sigma$ *for* $\mathcal{O}$*. That is,* $\mathcal{O}$ *consists of those sequences compatible with some* $\mu \in \mathcal{O}$ *of height* $n$*.*

*The partial order* $\leq$ *on* $S$ *is just the subset relation* $\subseteq$*.*

*A subset* $\mathcal{U}$ *of* $S$ **covers** $\mathcal{O} \in S$*,* $\mathcal{O} \lhd \mathcal{U}$*, if there is a finite* $n$ *such that, for all* $\mu \in T$ *of height* $n$*, either* $\mu \notin \mathcal{O}$ *or, for some* $\mathcal{O}_\mathcal{U} \in \mathcal{U}$ *of height at most* $n$*, we have* $\mathcal{O}_\mu \subseteq \mathcal{O}$ *and* $\mathcal{O}_\mu \subseteq \mathcal{O}_\mathcal{U}$*. In symbols,* $\mathcal{U}$ *covers* $\mathcal{O}$ *iff*

$$\exists n\, \forall \mu \in T\ ht(\mu) = n \to (\mu \notin \mathcal{O} \vee \exists \mathcal{O}_\mathcal{U} \in \mathcal{U}\ ht(\mathcal{O}_\mathcal{U}) \leq n \wedge \mathcal{O}_\mu \subseteq (\mathcal{O} \cap \mathcal{O}_\mathcal{U})).$$

*For any such* $n$*, we say that* $\mathcal{U}$ *covers* $\mathcal{O}$ *by height* $n$*.*

**Remark 8.** *By choosing the base to be empty,* $\emptyset \in S$*.*

*If the set of nodes compatible with* $\mu$ *is finite, then* $\mathcal{O}_\mu$ *is covered by the empty set: just choose* $n$ *to be large enough so that the tree beneath* $\mu$ *has died by height* $n$*. Hence it will represent the bottom element when the topology is viewed as a Heyting algebra.*

*For* $\mu \in T$ *and* $\mathcal{O} \in S$ *it is decidable from a base for* $\mathcal{O}$ *whether* $\mu \in \mathcal{O}$*.*

*Note that if* $\mathcal{U}$ *covers* $\mathcal{O}$ *by* $n$ *then* $\mathcal{U}$ *covers* $\mathcal{O}$ *by any* $k \geq n$*.*

**Proposition 9.** $(S, \leq, \lhd)$ *from above constitutes a formal topology.*

*Proof.* 1. Suppose $\mathcal{O} \in \mathcal{U}$; we need to show $\mathcal{U}$ covers $\mathcal{O}$. Let $\Sigma$ be a base for $\mathcal{O}$, witnessing that $n$ is the height of $\mathcal{O}$. As has already been remarked, for all $\mu$ of height $n$, it is decidable (from $\Sigma$) whether $\mu \in \mathcal{O}$ or not. If so, $\mathcal{O}_\mu \subseteq \mathcal{O}$, so $\mathcal{O}_\mathcal{U}$ can be chosen to be $\mathcal{O}$ itself.

2. Suppose $\mathcal{O}_1 \subseteq \mathcal{O}_0$ and $\mathcal{U}$ covers $\mathcal{O}_0$. We need to show $\mathcal{U}$ covers $\mathcal{O}_1$. We can assume that we have bases $\Sigma_0$ and $\Sigma_1$ for $\mathcal{O}_0$ and $\mathcal{O}_1$ respectively such that no $\mu_0 \in \Sigma_0$ extends any $\mu_1 \in \Sigma_1$. We can also assume that $\mathcal{U}$ covers $\mathcal{O}_0$ by height $n$. We will find a $k$ such that $\mathcal{U}$ covers $\mathcal{O}_1$ by $k$. Let $m$ be the height of the longest $\mu \in \Sigma_1$. Let $k$ be the larger of $m$ and $n$. Consider any $\mu$ of height $k$. If $\mu \notin \mathcal{O}_1$, then we are done. Else consider the initial segment $\rho$ of $\mu$ which is in $\Sigma_1$. Also consider $\mu \upharpoonright n \in \mathcal{O}_1$; recalling that $\mathcal{O}_1 \subseteq \mathcal{O}_0$, we conclude that $\mu \upharpoonright n \in \mathcal{O}_0$. By the choice of $n$, let $\mathcal{O}_\mathcal{U} \in \mathcal{U}$ be such that $\mathcal{O}_{\mu\upharpoonright n} \subseteq \mathcal{O}_\mathcal{U}$. Because $\mathcal{O}_\mu \subseteq \mathcal{O}_{\mu\upharpoonright n}$, $\mathcal{O}_\mu \subseteq \mathcal{O}_\mathcal{U}$, and we are done.

3. Suppose that $\mathcal{U}$ covers $\mathcal{O}$ by $n$, which we also take to be at least the height of $\mathcal{O}$, and that every $\mathcal{O}_\mathcal{U} \in \mathcal{U}$ is covered by $\mathcal{V}$. We need to show that $\mathcal{V}$ covers $\mathcal{O}$.

For each of the finitely many $\mu$'s of height $n$ that are in $\mathcal{O}$ let $\mathcal{O}_{\mathcal{U}\mu}$ be as given by the definition of covering. Each such $\mathcal{O}_{\mathcal{U}\mu}$ is covered by $\mathcal{V}$, which means there is a height $n_\mu$ as in the definition of covering, which we can also take to be at least as big as $ht(\mathcal{O}_{\mathcal{U}\mu})$ and $n$. Let $N$ be at least as big as each $n_\mu$. We would like to show that $\mathcal{V}$ covers $\mathcal{O}$ by $N$.

Let $\rho$ be of length $N$ and in $\mathcal{O}$. Letting $\mu$ be $\rho \upharpoonright n$ and using the $\mathcal{U}$ covering of $\mathcal{O}$ by $n \leq N$, we have $\mathcal{O}_{\mathcal{U}\mu}$ and $n_\mu$. By the $\mathcal{V}$ covering of $\mathcal{O}_{\mathcal{U}\mu}$, there is a member $\mathcal{O}_\mathcal{V}$ of $\mathcal{V}$ such that $\mathcal{O}_\rho \subseteq \mathcal{O}_\mathcal{V}$. Note that the height of $\mathcal{O}_\mathcal{V}$ is at most $n_\mu$, which is at most $N$.

4. Suppose $\mathcal{O}$ is covered by both $\mathcal{U}$ and $\mathcal{V}$. We need to show that $\mathcal{O}$ is covered by $\mathcal{W} = \{\mathcal{O}' \mid \exists \mathcal{O}_\mathcal{U} \in \mathcal{U}\ \mathcal{O}' \subseteq \mathcal{O}_\mathcal{U} \text{ and } \exists \mathcal{O}_\mathcal{V} \in \mathcal{V}\ \mathcal{O}' \subseteq \mathcal{O}_\mathcal{V}\}$.

We can assume that both $\mathcal{U}$ and $\mathcal{V}$ cover $\mathcal{O}$ by $n$, and will show that $\mathcal{W}$ covers $\mathcal{O}$ by $n$. Let $\mu \in \mathcal{O}$ have height $n$. Let $\mathcal{O}_\mathcal{U}$ be as given by $\mathcal{U}$ covering $\mathcal{O}$ – $\mathcal{O}_\mu \subseteq \mathcal{O}_\mathcal{U}$ – and $\mathcal{O}_\mathcal{V}$ be as given by $\mathcal{V}$ covering $\mathcal{O}$ – $\mathcal{O}_\mu \subseteq \mathcal{O}_\mathcal{V}$. Letting $\mathcal{O}' = \mathcal{O}_\mathcal{U} \cap \mathcal{O}_\mathcal{V}$ suffices. □

The reason for this formal topology is so that we can take the Heyting-valued model $\mathcal{M}_T$ over it.

# 4 The Kripke semantics

Now we specialize the general Kripke structure of constructive models from the earlier section to the case at hand, namely where $\phi(x)$ picks out the Heyting algebras $S$ induced by the formal topology on a decidable, infinite, binary tree $T$.[4]

What working with these particular Heyting algebras does for the semantics is that we can include heights (related to the heights of open sets) directly into the semantics, which are not available in general. The semantics defined earlier used a general notion of cover. In our context, this would allow for the definability of the solution to the halting problem, because information can be smuggled into realizers of membership in arbitrary covers, preventing a proof of the last theorem of this paper. That is why we have to work with a restricted notion of cover, one based on computability and containing less information.[5]

The covers we will use come in the form of membership functions, which are define via heights.

**Definition 8.** *A height of a node p:*

*For a Kripke node p, we define what it is for h, a tuple of natural numbers of the same length as p, to be a height of p, in notation $ht(p) \leq h$. For reasons that*

[4]This is the moment when the definition of the Kripke partial order switched from that of an earlier version, in which the inconsistent nodes were removed. Whether these nodes are present or removed makes no difference in a classical meta-theory. Within $\mathcal{M}_{K_1}$ though, the recursive realizability model, the consistency of a node is not a computably enumerable property, but rather co-c.e., inconsistency being so enumerable. To avoid having to do some sort of consistency check, we allow inconsistencies.

[5]When using the earlier semantics, although we don't have a proof of the last theorem, we also don't have a counter-example. It is an interesting question whether the following development could be taken as a instance of the earlier notion of cover. Perhaps the previous general semantics particularizes to the semantics about to be given.

*will be spelled out later, it is inconvenient to take ht to be a function, despite the suggestive notation $ht(p)$; rather, $ht(p) \leq h$ is a relation between $p$ and $h$, meant to convey the idea "if ht were a function then $ht(p)$ is componentwise less than or equal to $h$."*

*For the vacuous case of $p = \langle\rangle$, the only tuple of the same length is the empty tuple $\langle\rangle$ itself, which is a height of $p$. (It is unfortunate that in this case $p$ and $h$ are literally the same object. Whether $\langle\rangle$ is meant as a node $p$ or as a height $h$ should be clear from the context. Be that as it may, this occurs only in this degenerate case.)*

*For $p$ of length $n+1$, $p$ has height $h$ if $p \upharpoonright n$ has height $h \upharpoonright n$ and $p \upharpoonright n \Vdash_H \mathcal{O}^p_n$ has height at most $h(n)$.*

The idea is that $\mathcal{O}^p_0$ has height at most $h(0)$, and $p \upharpoonright 1 \Vdash \mathcal{O}^p_1$ has height at most $h(1)$, and so on. Note that nodes have both heights and lengths: as an $n$-tuple, $p$ has length simply $n$.

**Lemma 10.** *Every node $p$ has a height.*

*Proof.* By induction on the length of $p$. This is vacuous when the length is 0, and trivial when the length is 1, since $\mathcal{O}^p_0$ is an open set, which by definition has a base and hence a height. We show in detail the case of the length being 2, leaving the general case as a straightforward, albeit painful, exercise for the reader.

Since $\mathcal{O}^p_0 \Vdash_H$ "$\mathcal{O}^p_1$ is an open set", then $\mathcal{O}^p_0 \Vdash_H$ "$\mathcal{O}^p_1$ has a base and therefore a height". The issue is that no particular number may be forced to be a height. What we have is that $\mathcal{O}^p_0 \Vdash_H$ "$\exists h$ $h$ is a height of $\mathcal{O}^p_1$," which means that there is a cover $\mathcal{U}$ of $\mathcal{O}^p_0$ such that every member of $\mathcal{U}$ forces a particular integer to be a desired height. If $\mathcal{U}$ were finite, we could simply take the maximum of the integers so forced. But it may not be. But it doesn't have to be. By the definition of covering, $\mathcal{U}$ covers $\mathcal{O}^p_0$ by some height $n$. Go through the binary strings of height $n$. Those not in $\mathcal{O}^p_0$ can be ignored. The others form a finite, decidable set. Each of them induces a subset of some member of $\mathcal{U}$, which forces a particular height for $\mathcal{O}^p_1$. The maximum $h_1$ of those heights is therefore forced by $\mathcal{O}^p_0$ to be at least as big as $ht(\mathcal{O}^p_1)$. So the function such that $h(0) = n$ and $h(1) = h_1$ is a height for $p$.

□

**Remark 11.** *If we had instead defined $ht(p)$ as a function such that $p \upharpoonright n \Vdash_H ht(\mathcal{O}^p_n) = h(n)$ then $ht(p)$ might not always be defined, because different extensions of $p \upharpoonright n$ might force different heights for $\mathcal{O}^p_n$. Heights are taken to be tuples of actual natural numbers in the ground model, which is a mismatch with the components of $p$ which are in general just terms for open sets.*

*It is easy to see that the heights of $p$ are closed upwards (i.e. if $h$ is pointwise at least as big as some height of $p$ then $h$ is also a height of $p$).*

**Corollary 12.** *If $h_p$ is a height of $p$, and $q \geq p$, then there is a height $h_q$ of $q$ which (when restricted to the domain of $h_p$) is pointwise at least as big as $h_p$ (i.e. if $i$ is less than the length of $p$ then $h_q(i) \geq h_p(i)$).*

*Proof.* Take the pointwise maximum of $h_p$ and any height of $q$. □

In a sense, one could do away with heights as tuples and use simply the maximum of $h$'s components. But we're going to want to extend to tuples the idea of picking a height of the binary tree and going through all the strings at that height. This involves, for a pair say, having a height for the first tree, and for each string of that height, seeing what it tells us about other strings of that height, like whether they are in the second tree. The challenge is that strings on the first tree might have to be very long before they start to force facts about even short strings of the second tree. So if we took heights to be single natural numbers $h$, strings of height $h$ on the first tree might tell us little about strings of height $h$ on the second. That is why we allow heights to vary component by component.

**Definition 9.** *Heights of tuples and induced nodes:*

*For $\rho$ an $n$-tuple of binary sequences, the height of $\rho$ is the $n$-tuple of the heights of the components of $\rho$: $ht(\rho) = \langle ht(\rho(0)), ht(\rho(1)), \ldots, ht(\rho(n-1))\rangle$.*

*Each such $\rho$ induces a node $p_\rho$: $\mathcal{O}_i^{p_\rho} = \mathcal{O}_{\rho(i)}$. (Actually, this makes sense only when we have the topologies $\mathcal{T}_i$ at hand. So this notion will be used only in a context in which the $\mathcal{T}_i$'s are already determined.)*

We need to extend the notion of a binary sequence $\mu$ being a member of an open set $\mathcal{O}$ to tuples $\rho$. This starts to get complicated, because membership in $\mathcal{O}_i^p$ is not absolute, but only forced by $p \upharpoonright i$ or extensions thereof (of the same length). The definition of membership that suggests itself is that for all $i$ less than the length of $p$, $p_{\rho\upharpoonright i} \Vdash_H \rho(i) \in \mathcal{O}_i^p$. The issue with that is inconsistent nodes: if for instance $\mathcal{O}_{i-1}^p = \bot$ (in its Heyting algebra) then for sure $p_{\rho\upharpoonright i} \Vdash_H \rho(i) \in \mathcal{O}_i^p$, even when it is otherwise clear that $\rho(i)$ is not in $\mathcal{O}_i^p$. At the same time, $p_{\rho\upharpoonright i}$ will also force $\rho(i) \notin \mathcal{O}_i^p$, even in a setting in which $\rho(i)$ really is in $\mathcal{O}_i^p$.

The way we will deal with this situation is to give up on determinism. We must decide whether or not we want to consider $\rho(i)$ to be in $\mathcal{O}_i^p$, allowing that we may well decide we want both to take $\rho(i)$ to be in $\mathcal{O}_i^p$ and to take $\rho(i)$ not to be in $\mathcal{O}_i^p$.

**Definition 10.** *For $\rho$ of the same length $n$ as $p$, we say that $m_\rho$* **witnesses whether** *$\rho$* **is in** *$p$ if either:*

- *$m_\rho = 1$, and for all $i < n$, $p_{\rho\upharpoonright i} \Vdash_H \rho(i) \in \mathcal{O}_i^p$, in which case $m_\rho$ witnesses that $\rho$ is in $p$, or*
- *$m_\rho = \langle 0, j\rangle$ for some $j < n$, $p_{\rho\upharpoonright i} \Vdash_H \rho(i) \in \mathcal{O}_i^p$ for all $i < j$, and $p_{\rho\upharpoonright j} \Vdash_H \rho(j) \notin \mathcal{O}_i^p$, in which case $m_\rho$ witnesses that $\rho$ is not in $p$.*

*A* **membership function** *$m$* **for** *$p$ is a function with domain all tuples $\rho$ of some fixed height $h_m$ of $p$ such that $m(\rho)$ witnesses whether $\rho$ is in $p$; $h_m$ is called the* **domain height** *of $m$.*

*A height $h$ of $p$ is a* **canonical height** *if it is the height $h_m$ of some membership function $m$ for $p$.*

**Lemma 13.** *If $g$ is a height of $p$ then there is an $h$ pointwise at least as big as $g$ which is a canonical height of $p$.*

*Proof.* Again, we do the case in which the length in question is 2.

Membership in $\mathcal{T}_1$ is (forced to be) decidable. So membership in an open subset of $\mathcal{T}_1$ is decidable from a base for the open set. In particular, for any string $\mu$ of height $g(1)$, $\mathcal{O}_0^p \Vdash_H \mu \in \mathcal{O}_1^p \vee \mu \notin \mathcal{O}_1^p$. That means that $\mathcal{O}_0^p$ is covered by, say, $\mathcal{U}_\mu$, each member of which decides $\mu \in \mathcal{O}_1^p$ one way or the other. Furthermore, this covering happens by some height, say $n_\mu$, which we can take to be at least as big as $g(0)$. Letting $h(0)$ be the maximum of the $n_\mu$'s and $h(1)$ be $g(1)$, $h$ is as desired.

□

Membership functions will be used in the definition of the Kripke semantics. They will replace the arbitrary covers present in the general Kripke semantics from section 2.

**Definition 11.** *The restriction of a tuple to a height, $\rho \upharpoonright h$: Suppose $\rho$ is at least as long as $h$, and the height of $\rho$ is at least as big pointwise as $h$ beneath the length of $h$. (That is, for $i$ less than the length of $h$, the height of $\rho(i)$ is at least $h(i)$.) Then $\rho \upharpoonright h$ is the function with domain the length of $h$ such that $(\rho \upharpoonright h)(i) = \rho(i) \upharpoonright h(i)$.*

**Lemma 14.** *If $m_p$ is a membership function for $p$ with domain height $g$, and $q \geq p$, then there is a membership function $m_q$ for $q$ with domain height $h$ pointwise at least as big as $g$ such that $m_q$ extends $m_p$ in that, for all $\rho$ in the domain of $m_q$, if $m_p(\rho \upharpoonright g) = \langle 0, i\rangle$ then $m_q(\rho) = \langle 0, j\rangle$ for some $j \leq i$ .*

The idea is that once a node $p_{\rho\upharpoonright g}$ is seen to be inconsistent then any extension of it, say $\rho$, will also be declared inconsistent, and for a reason no bigger than the earlier reason. (We want to allow for $i$ to shrink because, for instance, it could be that $\rho(0)$ is a lot longer than $\rho \upharpoonright g(0)$, and so perhaps we can see that the former is not in $\mathcal{T}_0^p$ whereas the latter is.)

*Proof.* The construction in the previous lemma can begin from $m_p$. □

The nodes which $m$ evaluates to 1 form a cover for $p$, and so we have the following definition.

**Definition 12.** $P_m = \{\rho \in dom(m) \mid m(\rho) = 1\}$

By way of notation, the semantics about to be introduced will be written as simply $\Vdash$. It is to be distinguished from realizability semantics $\Vdash_r$, and the Heyting semantics $\Vdash_H$, and also from the earlier Kripke semantics $\Vdash_K$, even though the latter is what $\Vdash$ is most similar to.

**Definition 13.** *Implicitly in what follows, when we write "$p \Vdash \phi$", the parameters in $\phi$ are all in $\mathcal{M}^p$. Also implicit is the application of the transition function $f$, as need be.*

- *$p \Vdash \sigma \in \tau$ iff for some membership function $m$ of $p$ and for every $\rho \in P_m$ there is a $\sigma_\rho \in \tau(p_\rho)$ such that $p_\rho \Vdash \sigma = \sigma_\rho$.*

- *$p \Vdash \sigma = \tau$ iff for all $q \geq p$ and all $\upsilon \in \sigma(q)$, $q \Vdash \upsilon \in \tau$, and vice versa.*

- *$p \Vdash \phi \wedge \psi$ iff $p \Vdash \phi$ and $p \Vdash \psi$.*

- *$p \Vdash \phi \vee \psi$ iff for some membership function $m$ of $p$ and for every $\rho \in P_m$ either $p_\rho \Vdash \phi$ or $p_\rho \Vdash \psi$.*

- *$p \Vdash \phi \rightarrow \psi$ iff for all $q \geq p$ if $q \Vdash \phi$ then $q \Vdash \psi$.*

- *$p \Vdash \forall x\ \phi(x)$ iff for all $q \geq p$ and $\sigma \in \mathcal{M}^q$ $q \Vdash \phi(\sigma)$.*

- *$p \Vdash \exists x\ \phi(x)$ iff for some membership function $m$ of $p$ and for every $\rho \in P_m$ there is a $\sigma_\rho$ such that $p_\rho \Vdash \phi(\sigma_\rho)$.*

Recall that the symbol $\perp$ is taken to be 0=1, and is used implicitly in negations ($\neg\phi = \phi \rightarrow \perp$). If $p$ is an inconsistent node, then it can be shown inductively on $\phi$ that $p \Vdash \phi$ (because some membership function $m$ never takes on the value 1, so $P_m$ is empty).

**Lemma 15.** *Monotonicity: if $p \Vdash \phi$ and $q \geq p$ then $q \Vdash \phi$.*

*Proof.* By induction on $\phi$.

$\in$: For $m_p$ such a witnessing membership function for $p$ (with canonical height $g$), let $m_q$ be as given in the previous lemma. If $m_q(\rho) = 1$ then $m_p(\rho \restriction g) = 1$. Let $\sigma_\rho$ be $\sigma_{\rho \restriction g}$. Since $p_\rho \geq p_{\rho \restriction g}$, by induction $\sigma_\rho$ is as desired.

$=, \rightarrow, \forall$: Immediate, since the original definitions quantified over all extensions.

$\wedge$: Immediate by induction.

$\vee, \exists$: Similar to $\in$.

□

**Lemma 16.** *Each node satisfies the equality axioms.*

**Lemma 17.** *Each node satisfies constructive logic.*

*Proof.* Straightforward. □

**Theorem 18.** *This structure models IZF.*

*Proof.* For many of the axioms, we will merely give the construction of the witness, and leave it to the reader to check that the witness given satisfies the conditions listed above to be in the universe.

Empty Set: In the ground model $\mathcal{M}$, the constant function with domain the entire tree always returning the empty set is an object of rank 0, and represents the empty set.

Infinity: First, we show that each $n \in \omega$ is canonically represented by a set in the model, inductively on $n$. The case $n = 0$ is done by the empty set, above. Suppose for $i \leq n$, $i$ is represented in the model at the root by $\sigma_i$ of rank $i$. Then

$n+1$ will be represented by $\sigma_{n+1}$, where $\sigma_{n+1}(p)$ will be $\{f(\sigma_0), f(\sigma_1), ..., f(\sigma_n)\}$ (where once again $f$ is the (polymorphic) transition function). Finally, $\omega$ will be represented by $\sigma_\omega$, where $\sigma_\omega(p) = \{f(\sigma_n) \mid n \in \omega\}$.

Pair: If $\sigma, \tau \in \mathcal{M}^p$, then the pair is given by $\rho$, where $\rho(q) = \{f(\sigma), f(\tau)\}$.

Union: For $\sigma \in \mathcal{M}^p, \bigcup\sigma$ is given by $(\bigcup\sigma)(q) = \bigcup\{\tau(q) \mid \tau \in \sigma(q)\}$.

Extensionality: This follows fairly directly from the definition of forcing equality.

$\in$-Induction: What this comes down to is that all of the models are actually well-founded. That enables us to use $\in$-Induction from the ground model. The challenge is that if $p \Vdash \sigma \in \tau$, as sets in the ground model $\sigma$ is not actually a member of $\tau$; rather, at best, $\sigma \in \tau(p)$. We still have that $\sigma$ occurs before $\tau$, in the sense that $\sigma$ is in the transitive closure of $\tau$. This suggests using a stronger form of induction, namely induction on the transitive closure of a set. This is like the difference between strong and weak induction on the natural numbers. In weak induction, the inductive hypothesis is $\phi(n)$, from which one is to derive $\phi(n+1)$; in contrast, in strong induction, one is to "remember" all previous values, in that the inductive hypothesis is $\forall k < n\ \phi(k)$, from which one is to derive $\phi(n)$.

In more detail, suppose $p \Vdash \forall x\ ((\forall y \in x\ \phi(y)) \to \phi(x))$. We must show that $p \Vdash \forall x\ \phi(x)$.

Work in the ground model. Given a term $\sigma$ from some $\mathcal{M}^q$, $q$ is uniquely determined by $\sigma$, since $\sigma$ is a function with domain the set of extensions of $q$. Therefore we can use the notation $q_\sigma$ to indicate this $q$. Let $\psi(x)$ be "for every member of the transitive closure of $\{x\}$ which is a term $\sigma$ with $q_\sigma \geq p$, $q_\sigma \Vdash \phi(\sigma)$." We will show $\forall x\ \psi(x)$ by $\in$-Induction. Toward that end, assume $\psi(y)$ for all $y \in x$. We must show $\psi(x)$. Let $z$ be in the transitive closure of $\{x\}$, and also be such a term, with $q_z \geq p$. Either $z \in TC(\{y\})$ for some $y \in x$ or $z = x$. In the former case, by the inductive hypothesis we are done. In the latter case, notice that $p \Vdash \forall y \in x\ \phi(y)$, because any such $y$ is (forced to be equal to something) in the transitive closure of some member of $x$. Using the main inductive hypothesis, $p \Vdash \phi(x)$. We have just shown that if $\psi(y)$ for all $y \in x$ then $\psi(x)$. By $\in$-Induction applied to $\psi$, we have that for all $x\ \psi(x)$.

Since $x \in TC(\{x\})$ in particular, if $x$ is a term with $q_x \geq p$ then $q_x \Vdash \phi(x)$. In other words, $p \Vdash \forall x\ \phi(x)$, as was to be shown.

Power Set: Let $\sigma \in \mathcal{M}^p$. Then let $(\wp(\sigma))(q)$ be $\{\tau \in \mathcal{M}^q \mid \forall r \geq q\ \tau(r) \subseteq \sigma(r)\}$.

Separation: Given $\sigma \in \mathcal{M}^p$, and $\phi(x)$ with parameters from $\mathcal{M}^p$, let $(\text{Sep}(\sigma, \phi))(q)$ be $\{\rho \in \sigma(q) \mid q \Vdash \phi(\rho)\}$.

Collection: Suppose $p \Vdash \forall x \in \sigma\ \exists\tau\ \phi(x, \tau)$. Working in the ground model $\mathcal{M}$, for each $q \geq p$ and $\rho \in \sigma(q)$, there is an $\alpha$ such that some $\tau$ in $\mathcal{M}^q_\alpha$ satisfies $q \Vdash \phi(\rho, \tau)$. By Collection in $\mathcal{M}$, there is a bounding set for all of the $\alpha$'s needed. This bounding set can be restricted to contain only ordinals, and then expanded to be an ordinal itself, say $\beta$. Let $\Upsilon$ be such that, for $q \geq p$, $\Upsilon(q) = \mathcal{M}^q_\beta$, which suffices for a bounding set.

Actually, more can be said here. In ZF set theory, over the other ZF axioms, Replacement, Collection, and Reflection are equivalent. Within the context of

IZF though, over the other IZF axioms, Reflection implies Collection, which itself implies Replacement, yet both of those implications apparently[6] do not reverse. Often Replacement is more difficult to work with. For instance, in our context, $p$ might force $\phi$ to be a function, so $p$ would force that the $y$ such that $\phi(x, y)$ is unique – but the choice of a term to stand for such a $y$ is not! So using Replacement in the meta-theory to get Replacement in the model seems hopeless.[7] Typically one can and does use Collection or Reflection in the meta-theory to get the same axiom holding in the model. Reflection, albeit the strongest of these principles, is usually the easiest to show. The assertion $\psi$ that some statement $\phi$ you would like to reflect is true within the Kripke model is itself some assertion, even if more complex than $\phi$, in the ground model. One uses Reflection in the ground model to get a set $X$ in which $\psi$ holds. Then one takes the Kripke model as built within $X$. Because $\psi$ holds in $X$, $\phi$ holds in $X$'s version of the Kripke model, and because $X$ is a set, its version of the Kripke model is merely a set within the full Kripke model. (If you want to know why Reflection is true in the ground model, one can consider the ground model as being built within a classical meta-theory, and use Reflection in $V$ to get Reflection in the ground model, in a similar fashion as above.)

□

## 5 FAN$_\Delta$ does not imply FAN$_c$

First we will show that the model satisfies FAN$_\Delta$, and then that is falsifies FAN$_c$.

By way of showing the former, we first indicate why forcing with $S_T$, $T$ the complement of $B$, eliminates $B$ as a non-uniform bar. Then we show how the construction strings these various forcings together to eliminate all such possible $B$'s, resulting in FAN$_\Delta$ being true.

**Theorem 19.** *In $\mathcal{M}_T$, the generic $G$ is (identifiable with) an infinite branch through $T$.*

*Proof.* We can identify the generic $G$ with $\{\langle \mathcal{O}_\mu, \nu \rangle \mid \nu \subseteq \mu, \mathcal{O}_\mu$ a basic open set$\}$. Essentially all that needs showing is that $\mathcal{O}_\emptyset \Vdash_H$ "for all $k$ there is a unique $\nu$ of height $k$ with $\nu \in G$." (Of course, $G$ being a branch also needs that the various $\nu$'s also cohere with one another. But that's easy: if $\perp \neq \mathcal{O}_\mu \Vdash \nu \in G$, then $\mu$ and $\nu$ are compatible. (Note that if $\mu$ has a unique child in $T$, then $\nu$ could actually be longer than $\mu$.) So if two $\nu$'s are forced to be in $G$, choose $\mu$ longer than both $\nu$'s forcing those facts; then both $\nu$'s are initial segments of $\mu$, hence compatible with one another.)

[6]That is, it is known that Replacement does not imply Collection; see [8]. It is known that Collection does not imply Reflection over CZF + Separation, which is close to IZF, so presumably this non-implication is the case also over IZF, although that has yet to be proven; see the end of [12] for an example and discussion.

[7]This is a known phenomenon, already having occurred elsewhere. For an example, see [11].

Since the natural numbers in the sense of $\mathcal{M}_T$ can be identified with those of the meta-theory, it suffices to fix a $k$ in the sense of $V$, and show that $\mathcal{O}_\emptyset \Vdash_H$ "there is a unique $\nu$ of height $k$ with $\nu \in G$." It is easy to see that if $\mathcal{O}_\mu$ is a basic open set with $\mu$ of height $k$ then $\mathcal{O}_\mu \Vdash_H$ "$\mu$ is the unique member of $G$ of height $k$." More to the point, $\mathcal{O}_\mu \Vdash_H$ "There is a unique member of $G$ of height $k$." Let $\mathcal{U}$ be $\{\mathcal{O}_\mu \mid \mu \text{ has height } k\}$. It is trivial to show that $\mathcal{U}$ covers $\mathcal{O}_\emptyset$ by height $k$. By the semantics of Heyting-valued models, $\mathcal{O}_\emptyset$ forces the same. (Notice that we are not in the Kripke model, so we are not to use the Kripke-esque semantics given above. Rather, the claim is about a Heyting extension.) □

**Corollary 20.** *For $p$ a node in the Kripke partial order, with final entry $(T, \mathcal{O})$, and $B$ (a term for) the complement of $T$, $p \Vdash B$ is not a bar.*

*Proof.* Let $G$ be the generic for forcing with $T$. The function (with domain the partial order from $p$ onwards) with constant output $G$ (more accurately, the canonical image of $G$ in the input's associated model) witnesses that $B$ is not a bar. □

The next two theorems finish this paper. This is the point at which we must work in $\mathcal{M}_{K_1}$.

**Theorem 21.** $\langle\rangle \Vdash FAN_\Delta$.

*Proof.* The idea is simple enough. If, at a node, $B$ is forced to be a decidable bar, then $B$ must also be forced to be uniform, because, if not, the node would have an extension given by forcing with the complement of $B$, showing that $B$ could not have been a bar. We need to check the details though, to guard against things like the use of classical logic and to make sure we're using the semantics of the model at hand. For better or worse, I know of no other way to do this than to unravel the statement to be shown, using the semantics given.

We need to show $\langle\rangle \Vdash \mathrm{FAN}_\Delta$, working within $\mathcal{M}_{K_1}$, meaning we must find a realizer $e$ for the statement $\langle\rangle \Vdash \mathrm{FAN}_\Delta$. As a reminder, $\langle\rangle$ is the empty sequence, the bottom node in the partial order underlying the model. For reference, $\mathrm{FAN}_\Delta$ is the assertion "for all $B$, if $B$ is an upwards-closed decidable bar (in $2^{<\omega}$), then $B$ is uniform, i.e. there is a natural number $n$ such that all binary sequences of height $n$ are in $B$."

Unpacking the meaning of $\Vdash$, we need to show that within $\mathcal{M}_{K_1}$, if $B \in \mathcal{M}^p$ then $p \Vdash$ "if $B$ is such a bar then $B$ is uniform." That means that

$$\text{if } t \Vdash_r \text{ "}p \text{ is a node and } B \in \mathcal{M}^p\text{" then } \{e\}(t) \Vdash_r \text{ "}p \Vdash \text{(if } B \text{ is such a bar then } B \text{ is uniform)."} \tag{1}$$

To save on notation, we will suppress mention of $t$. This means that we must show

$$e \Vdash_r \text{ "for all } q \geq p, \text{ if } q \Vdash B \text{ is such a bar then } q \Vdash B \text{ is uniform."} \tag{2}$$

Again suppressing the realizer that $q \geq p$, we must show

$$e \Vdash_r \text{“if } q \Vdash B \text{ is such a bar then } q \Vdash B \text{ is uniform.”} \tag{3}$$

So, suppose $f \Vdash_r$ “$q \Vdash B$ is such a bar;” we must have that $\{e\}(f) \Vdash_r$ “$q \Vdash B$ is uniform.”

**The Hypothesis:** We will have occasion to refer to realizers computable from $f$ (and from other functions too). In all such cases there will be a specific construction of the derived realizer, but there is little reason to be so explicit. So we use the notation $f^*$ in such cases. For instance, there is a realizer $g$, easily computable from $f$, with $g \Vdash_r$ “$q \Vdash B$ is decidable;” we will use instead the notation $f^* \Vdash_r$ “$q \Vdash B$ is decidable.” Unpacking that yields

$$f^* \Vdash_r \text{“}q \Vdash \text{ (for all } \mu \in 2^{<\omega}, \text{ either } \mu \in B \text{ or } \mu \notin B).\text{”} \tag{4}$$

Since $2^{<\omega}$ does not change from node to node, that means

$$f^* \Vdash_r \text{“for all } \mu \in 2^{<\omega}, q \Vdash \text{ (either } \mu \in B \text{ or } \mu \notin B).\text{”} \tag{5}$$

Identifying a realizer that $\mu \in 2^{<\omega}$ with $\mu$ itself, that becomes

$$\text{for all } \mu \in 2^{<\omega},\ f^*\mu \Vdash_r q \Vdash (\mu \in B \vee \mu \notin B). \tag{6}$$

(The notation $f^*\mu$ means it is uniformly computable from $f$ and $\mu$.) And that means that

$$\begin{aligned} \text{for all } \mu \in 2^{<\omega}, f^*\mu \Vdash_r \text{“there is an } m_\mu \text{ such that} \\ m_\mu \text{ is a membership function for } q \\ \text{and for each } \rho \in P_{m_\mu} \text{ either } p_\rho \Vdash \mu \in B \text{ or } p_\rho \Vdash \mu \notin B.\text{”} \end{aligned} \tag{7}$$

Unpacking further,

$$\begin{aligned} \text{for all } \mu \in 2^{<\omega} \text{ there is an } m_\mu \text{ such that} \\ f^*\mu \Vdash_r \text{“}m_\mu \text{ is a membership function for } q \\ \text{and for each } \rho \in P_{m_\mu} \text{ either } p_\rho \Vdash \mu \in B \text{ or } p_\rho \Vdash \mu \notin B.\text{”} \end{aligned} \tag{8}$$

In order to realize that $m_\mu$ is a membership function, it must be realized that $m_\mu$ is a function with domain determined by $h_{m_\mu}$; in other words, $f$ and $\mu$ together computes $m_\mu$ and $h_{m_\mu}$. To bring that out in the notation, instead of $m_\mu$ we will write $m_{f^*\mu}$.

$$\begin{aligned} f^*\mu \Vdash_r \text{“}m_{f^*\mu} \text{ is a membership function for } q \\ \text{and for each } \rho \in P_{m_{f^*\mu}} \text{ either } p_\rho \Vdash \mu \in B \text{ or } p_\rho \Vdash \mu \notin B.\text{”} \end{aligned} \tag{9}$$

**The Conclusion:** Having just unpacked the hypothesis, we will now analyze the conclusion. Recall what we need to show: $\{e\}(f) \Vdash_r$ “$q \Vdash B$ is uniform;” i.e. $\{e\}(f) \Vdash_r$ “$q \Vdash$ there is a bound $n$ witnessing that $B$ is uniform;” which is

$$\begin{aligned} \{e\}(f)^* \Vdash_r \text{“there is a membership function } m \text{ for } q, \\ \text{and for all } \rho \in P_m \text{ there is some object } n \text{ such that} \\ p_\rho \Vdash (n \text{ is a natural number witnessing the uniformity of } B).\text{”} \end{aligned} \tag{10}$$

That comes down to the computability from $\{e\}(f)$ of a function $m$ such that

$$\begin{aligned} \{e\}(f)^* \Vdash_r \text{"}m \text{ is a membership function for } q, \\ \text{and for all } \rho \in P_m \text{ there is some object } n \text{ such that} \\ p_\rho \Vdash (n \text{ is a natural number witnessing the uniformity of } B)\text{."} \end{aligned} \tag{11}$$

Since this is complicated enough, we're now going to build up somewhat slowly. We will examine several cases, based on the length of $q$. Since $e$ has access to a realizer that $q$ is a node, $e$ has access to $q$'s length, and so can make this case distinction.

**Case I:** For the first pass, suppose $q = \langle\rangle$. The only sequence of length 0 is the empty sequence, so the only possible membership function for $q$ is the $m$ with domain $\{\langle\rangle\}$. Furthermore, it could not be the case that $m(\langle rangle) = \langle 0, j\rangle$, since $j$ would have to be less than $n$, which is 0, so $m(\langle\rangle) = 1$. Hence, by the hypothesis, we have $f^*\mu \Vdash_r$ "either $\langle\rangle \Vdash \mu \in B$ or $\langle\rangle \Vdash \mu \notin B$." Similarly for the conclusion: we must have $\{e\}(f) \Vdash_r$ "there is some $n$ such that $\langle\rangle \Vdash n$ is a natural number witnessing the uniformity of $B$." The obvious algorithm to find a uniform bound for $B$ is to go through the various binary sequences $\mu$ in some order so that a sequence is examined before any longer sequence; compute $f^*\mu$; see whether $f^*\mu$ realizes $\langle\rangle \Vdash \mu \in B$ or $\langle\rangle \Vdash \mu \notin B$; continue until it finds a height $n$ with all sequences of that height forced into $B$: the result of that computation suffices. All there is left to do in this case is to show that this algorithm terminates. If not, then every $K_1$ realizer will realize that $B$ is not uniform. So, in $\mathcal{M}_{K_1}$, letting $T$ be the complement of $B$, $\langle(S_T, \mathcal{O}_{\langle\rangle})\rangle$ is a node (where $S_T$ is the formal topology induced by $T$). (Actually, there is a subtlety here. $B$ is a set in the Kripke model. In order for $S_T$ to give a node, $B$ would have to be a set in $\mathcal{M}_{K_1}$ instead. This is not a real problem though. Using the decidability of $B$ in $\mathcal{M}$, a set can be built in $\mathcal{M}_{K_1}$ which has the same members as $B$ is forced to have. By abuse of notation, we call this $\mathcal{M}_{K_1}$ set $B$ also.) By the corollary, $\langle(S_T, \mathcal{O}_{\langle\rangle})\rangle \Vdash$ "$B$ is not a bar," contradicting the assumption that $f \Vdash_r$ "$\langle\rangle \Vdash B$ is (such) a bar."

**Case II:** For our second pass, suppose that $q$ has length 1: $q = \langle(S_T, \mathcal{O})\rangle$. Consider the following algorithm. Go through the binary sequences $\mu$ (shorter before longer). Computer $m_{f^*\mu}$. Go through the members of $P_{m_{f^*\mu}}$ and see if they force $\mu$ in or out of $B$. Occasionally you might find a sequence $\rho \in \mathcal{O}$ forcing $B$ to be uniform. (That would be a sequence such that, for some $n$, every binary sequence of height $n$ is forced to be in $B$ by $p_\rho$; of course, even $\mu$'s of the same height might have different $m_{f^*\mu}$'s and $h_{m_{f^*\mu}}$'s, so the information that $\mu \in B$ might have been presented to us not via $p_\rho$ but rather $p_{\rho'}$ for some proper initial segment $\rho'$ of $\rho$, but no matter.) Keep track of those $\rho$'s. Also keep track of the extensions of those $\rho$'s in $\mathcal{O}$. If at any time all of the members $\rho$ in $\mathcal{O}$ of some fixed height are seen to force $B$ to be uniform, then that provides a membership function for $q$ as desired, with the history of that computation as the witness. It bears observation that it is necessary to consider not just the $\rho$'s presented to us via the $m_{f^*\mu}$'s, but also their extensions. That's because the $m_{f^*\mu}$'s might have domains all bounded at a level not forcing $B$ to be uniform but the troublesome nodes in $\mathcal{O}$ are all on dead ends. (Example: Suppose $\mathcal{O}$ contains $\langle 0 \rangle$ but no extensions of it. Let $p_{\langle 0 \rangle} \Vdash B = \emptyset$ and $p_{\langle 1 \rangle} \Vdash B = 2^{<\omega}$. Then you can find out everything you want to know about membership in $B$ by going through $\rho$'s of height 1, and if that's all you do, you will never find an entire level of $\mathcal{O}$ forcing $B$ to be uniform.)

All that we have to show for this case is that the algorithm just sketched terminates. Suppose not. We are of course aiming at a contradiction. The contradiction will be with $q$ forcing $B$ to be a bar. It would be nice if we could force with the complement of $B$ to generate a path missing $B$. But in order to be allowed to do that, $B$ would have to be non-uniform, and there is no reason to think $q$ forces $B$ not to be uniform. Some nodes in $q$ may well force $B$'s uniformity. In fact, the nodes in $\mathcal{O}$ forcing $B$ to be uniform could even be dense in $\mathcal{O}$, so maybe even no open subset of $\mathcal{O}$ forces $B$ not to be uniform. That means we first have to extend $q$, via a sub-forcing of $q$'s component $S_T$, to get $B$ to be non-uniform, and then extend again to get the desired path.

So let's return to the algorithm from above. Let's specify its operation in a bit more detail. At stage $n$, consider all $\mu$'s of height $n$. Calculate each $h_{m_{f^*\mu}}$, which is effectively a natural number (actually, a sequence of length 1 of a number), and call the maximum $h_n$. Without loss of generality, $h_n$ is an increasing function of $n$. Go through the members $\rho$ of $\mathcal{O}$ of height $h_n$. If $p_\rho$ is seen to force $B$ to be uniform, put $\rho$ (and all of its extensions) into a new set we're defining, $B_U$. Also any $\rho$ of height $h_n$ not in $\mathcal{O}$ is to be put into $B_U$. Else do not put $\rho$ into $B_U$. This defines $B_U$ within $\mathcal{M}_{K_1}$. It is easy to see that $B_U$ is decidable (as all $\rho$'s of height $h_n$ are decided at stage $n$, and $h_n \geq n$), and upwards closed. The hypothesized non-termination of the algorithm means exactly that $B_U$ is not uniform. Of course, the decidability, upwards closure, and non-uniformity of $B_U$ are all in $\mathcal{M}_{K_1}$, but they carry over to $\mathcal{M}_q$. (Strictly speaking, when working within $\mathcal{M}_q$ we must refer to the canonical image of $B_U$, but for notational convenience we will use the same name for such an image as for the original set.) So, letting $T_U$ be the complement of $B_U$, there is an extension $r = q^\frown(S_{T_U}, \mathcal{O}_{\langle\rangle})$ of $q$.

We will need some facts about $T_U$ and about forcing with $T_U$ later, which we may as well establish now. For one, $T_U \subseteq T$, actually $T_U \subseteq \mathcal{O}$. For another, because $T_U \in \mathcal{M}_{K_1}$, many fact about $\mathcal{M}_r$ don't depend on the $S_T$ forcing at all, from $r$'s zeroth component, but just on $S_{T_U}$. In detail, an extension $\langle (S_T, \mathcal{O}_\kappa), (S_{T_U}, \mathcal{O}_\lambda) \rangle$ of $r$ might force $\sigma \in \tau$: $\mathcal{O}_\kappa \Vdash_{S_T}$ "$\mathcal{O}_\lambda \Vdash_{S_{T_U}} \sigma \in \tau$", where $\Vdash_{S_T}$ and $\Vdash_{S_{T_U}}$ are Heyting algebra forcings, earlier called $\Vdash_H$; the notational change is just to make the Heyting algebra explicit. If $\sigma$ and $\tau$ are terms in $\mathcal{M}_{K_1}$ for forcing with $S_{T_U}$, then the contribution of $\mathcal{O}_\kappa$ is irrelevant, and the earlier assertion is simply equivalent with "$\mathcal{O}_\lambda \Vdash_{S_{T_U}} \sigma \in \tau$". Another way to look at this is that since $T_U$ is in $\mathcal{M}_{K_1}$, $r$ does not represent an iterated forcing so much as a product forcing.

We would like to extend $r$ by forcing with the complement of $B$, rightly interpreted, because that would then show that $B$ is not a bar in some extension of $q$, the desired contradiction. The idea here is that $S_{T_U}$ is a subset of $\mathcal{O}$, so $B$, as a term in the language for forcing with $S_T$, can be restricted to a term $B^\restriction$ in the language for $S_{T_U}$. In a bit of detail, a term consists of pairs $\langle \mathcal{O}_\sigma, \sigma \rangle$, where $\mathcal{O}_\sigma$ is an open set and $\sigma$ is (inductively) a term. In the case of $B$, because $B$ is forced to be a set of binary strings, without loss of generality we can take $\sigma$ to be a canonical term for a binary string $\mu$, which by abuse of notation we will also call $\mu$. (More generally, $\sigma$ might merely be a term for such a sequence, partially unspecified, which needs extensions of $q$ to become fully specified. To handle this more general situation, the construction of $B^\upharpoonright$ from $B$ should be understood as working not just on the members of $B$ but also hereditarily.) Also, the open set $\mathcal{O}_\sigma$ can always be taken to be a basic open set $\mathcal{O}_\nu$ ($\nu$ a binary string), for conceptual simplicity. If $\langle \mathcal{O}_\nu, \mu \rangle \in B$ and $\nu \in S_{T_U}$, then place $\langle \mathcal{O}_\nu, \mu \rangle$ into $B^\upharpoonright$. If in contrast $\nu \notin S_{T_U}$, then do not place $\langle \mathcal{O}_\nu, \mu \rangle$ into $B^\upharpoonright$. The idea here is that $B^\upharpoonright$ is to act like $B$ if the forcing were with $T_U$ instead of $T$. Of course, we could achieve the same result by interpreting $B$ within $S_{T_U}$. But we will in contrast want to see, within $S_T$, how $B$ would behave under $S_{T_U}$. That is the purpose of $B^\upharpoonright$: it behaves when forcing with $S_T$ the way $B$ behaves when forcing with $S_{T_U}$.

The following facts should be brought out. Despite their tightness with each other, $B$ and $B^\upharpoonright$ can be forced by extensions of $r$ to look very different from one another, as the former is determined by forcing with $S_T$ and the latter by $S_{T_U}$, two independent forcings. That much being understood, both terms can be interpreted under each forcing. There we do see relations, because when forcing with one Heyting algebra $H$ both $B$ and $B^\upharpoonright$ are interpreted under the same generic. For one, $S_{T_U} \Vdash_H B = B^\upharpoonright$. For another, because $B^\upharpoonright$ is literally a subset of $B$, any Heyting algebra will force "$B^\upharpoonright \subseteq B$". For a third, if $\nu \in T_U$, and $\mathcal{O}_\nu \Vdash_{S_T} \mu \in B$, then $\mathcal{O}_\nu \Vdash_{S_{T_U}} \mu \in B^\upharpoonright$. Perhaps surprisingly, the converse to that does not hold! It could be that $\nu$ leads to a dead-end in $T_U$, so $\mathcal{O}_\nu$ forces everything there, whereas it doesn't in $T$.

We want to show that $r \Vdash_H B^\upharpoonright$ is decidable, upwards closed, and non-uniform. Once we do that, we will be able to extend $r$ by forcing with the complement of $B^\upharpoonright$, which we will call $T^\upharpoonright$, and can then show our contradiction. The decidability and upwards closure are inherited from $q$ and $B$. As for the non-uniformity, suppose that $r' \leq_H r$ is such that $r' \Vdash_H B^\upharpoonright$ is uniform. We must show $r' =_H \bot$. For conceptual simplicity, and without loss of generality, assume $r'$ is given by basic open sets: $r' = \langle (S_T, \mathcal{O}_\kappa), (S_{T_U}, \mathcal{O}_\lambda) \rangle$. By the discussion above, $\mathcal{O}_\lambda \Vdash_{S_{T_U}} B^\upharpoonright$ is uniform. With a bit of work, we can get particular natural numbers $n$ and $k$ such that every extension $\iota$ of $\lambda$ in $T_U$ of height $n$ forces (in the sense of $S_{T_U}$) $B^\upharpoonright$ to be uniform by $k$ (meaning each string of height $k$ is forced to be in $B^\upharpoonright$); moreover, the witness to any $\xi$ of length $k$ being in $B^\upharpoonright$ is some initial segment of $\iota$ (i.e. $\langle \mathcal{O}_{\iota'}, \xi \rangle \in B^\upharpoonright$ for some initial segment $\iota'$ of $\iota$). Since $B^\upharpoonright$ is literally a subset of $B$, the same holds for $B$ in place of $B^\upharpoonright$.

What would our algorithm for building $B_U$ and $T_U$ have done with such an $\iota$? Possibly when checking the realizability of "$p_\iota \Vdash \xi \in B \vee p_\iota \Vdash \xi \notin B$" ($\xi$ of length $k$), the first option would always have been chosen (that is, for all $\xi$), so $\iota$ would have been placed into $B_U$, and is not even in $T_U$. It is entirely possible though that the realizer gave us the second option. Since the first option still remains true, that means that $p_\iota$ is inconsistent. In conclusion, for every extension $\iota$ of $\lambda$ of length $n$ in $T_U$, $p_\iota =_H \bot$. Hence $\mathcal{O}_\lambda = \bot$, and $r' = \bot$.

Therefore $s := r^\frown\langle(S_{T^\upharpoonright}, \mathcal{O}_{\langle\rangle})\rangle$ is an extension of $r$, and in $\mathcal{M}_s$, by the previous theorem the generic $G$ is an infinite branch through $T^\upharpoonright$. Since $q \Vdash B$ is a bar and $s \geq q$, $s \Vdash$ there is a node in $G \cap B$. Let $m$ be a canonical height of $s$ such that for every $\rho \in P_m$ $p_\rho$ forces a particular member of $G \cap B$. As a reminder, such a $\rho$ is a triple $\langle \rho_T, \rho_{T_U}, \rho_{T^\upharpoonright}\rangle$. Because $T_U \subseteq \mathcal{O}$, there is a $\rho$ with $\rho_T \in T_U$ and compatible with $\rho_{T_U}$. Fix such a $\rho$, and let $p_\rho \Vdash \mu \in G \cap B$. Because $p_\rho \Vdash \mu \in G$, $p_\rho \Vdash \mu \in T^\upharpoonright$. So $p_\rho \Vdash \mu \notin B^\upharpoonright$. On the other hand, $p_\rho \Vdash \mu \in B$. But since $\rho_T$ and $\rho_{T_U}$ are compatible, no disagreement can be forced between $B$ and $B^\upharpoonright$, for our contradiction.

**Case III:** It is time to finish the proof of this theorem. Most of the work was done in Case II, so we will be somewhat sketchy.

Let $q$ be a node of length at least 1. To recall, we have a realizer $f$ that $q \Vdash B$ is an upwards closed decidable bar. We are searching for a computable membership function for $q$ and a realizer that everything the membership function evaluates to 1 forces $B$ to be uniform. We will mimic the argument in Case II as much as possible.

Go through the binary sequences $\mu$ (shorter before longer). Computer $m_{f^*\mu}$. Go through the members of $P_{m_{f^*\mu}}$ and see if they force $\mu$ in or out of $B$. Occasionally you might find a member $\rho$ of $q$ forcing $B$ to be uniform. Keep track of those $\rho$'s. Also keep track of the extensions of those $\rho$'s that are members of $q$. If at any time all of the members $\rho$ of $q$ of some fixed canonical height are seen to force $B$ to be uniform, then that is a height for $q$ as desired, with the history of that computation as the witness. It bears observation that it suffices to force the uniformity of $B$ by all members of $q$ of a fixed height, as then any pointwise larger canonical height of $q$ will be as desired.

All that we have to show is that the algorithm just sketched terminates. Suppose not, toward a contradiction with $q$ forcing $B$ to be a bar. It would be nice if we could force with the complement of $B$ to generate a path missing $B$. But in order to be allowed to do that, $B$ would have to be non-uniform, and there is no reason to think $q$ forces $B$ not to be uniform. That means we first have to extend $q$ to get $B$ to be non-uniform, and then extend again to get the desired path.

Returning to the algorithm from above in more detail, at stage $n$, consider all $\mu$'s of height $n$. Calculate each $h_{m_{f^*\mu}}$, and call the pointwise maximum, also a canonical height, $h_n$. Without loss of generality, $h_n$ is a pointwise increasing function of $n$. Go through the members $\rho$ of $q$ of height $h_n$. If $p_\rho$ is seen to force $B$ to be uniform, put $\rho$ (and all of its extensions) into a new set we're defining, $B_U$. Also any $\rho$ of height $h_n$ not a member of $q$ is to be put into $B_U$. Else do not put $\rho$ into $B_U$. This defines $B_U$ within $\mathcal{M}_{K_1}$. It is easy to see that $B_U$ is decidable (as all $\rho$'s of height $h_n$ are decided at stage $n$, and each component of $h_n$ is at least $n$), and upwards closed. The hypothesized non-termination of the algorithm means exactly that $B_U$ is not uniform, in the sense that for no height $h$ of $q$ is every tuple of binary sequences of height $h$ in $B_U$. Of course, the decidability, upwards closure, and non-uniformity of $B_U$ are all in $\mathcal{M}_{K_1}$, but they carry over to $\mathcal{M}_q$.

The current $B_U$ is different from the one in Case II, in that before it consisted simply of binary strings, and here it is a tuple of binary strings. So what we actually have to do is to decompose $B_U$ into its components $B_{U0}, B_{U1}, ..., B_{U(n-1)}$, where $n$ is the length of $q$. Then we extend $q$ via an iterated forcing with the complements of the $B_{Ui}$'s. After that, the argument continues pretty much as in the previous case. We extend again with the complement of $B$ as interpreted by this latter iterated forcing, what was called $B^{\upharpoonright}$ above. Finally, we get a contradiction as above with forcing a particular member of this last generic $G$, a path through $T^{\upharpoonright}$, into $B$.

□

**Theorem 22.** $\langle\rangle \Vdash \neg FAN_c$.

*Proof.* Recall that a $c$-fan is based on a decidable set of $C$, which can be taken to be a computable assignment of "in" and "out" to all the nodes. A node is in the bar if it and all of its successors are assigned "in", and out of the bar, or in the tree, if one of its successors is "out".

Consider the following $c$-fan, due to Francois Dorais. Let $K$ be some complete c.e. set, with enumeration $K_s$ ($K$ at stage $s$). We identify $K_s$ with its characteristic function up to $s$, so that $K_s$ is a binary sequence of height $s$. Let $C$ be such that all nodes on level $n$ are labeled "in" except for $K_n$, which is labeled "out". Since the integers in any $\mathcal{M}^p$ are the same as those in any $\mathcal{M}_p$, which are those of $V$, the set $C$ exists in the model as a decidable set, and is the internalization of $C$ as defined in $V$. Letting $\chi(n)$ be the characteristic function of $K \cap n$, $\chi$ is the unique branch missing the induced $c$-set $B$. As is well known, $\chi$ does not exist as a path in $\mathcal{M}_{K_1}$, so $B$ is a bar there. (Briefly, if $e \Vdash_r \chi$ is a total function, then $\chi(n+1)(n)$ is computable from $e$ and so $e$ provides a computable solution to the halting problem.) We must, and need only, show that $B$ remains a bar in $\mathcal{M}$, the idea being that generically $\chi$ is not added by forcing to the model.

Work in $\mathcal{M}_{K_1}$. Suppose that $p \Vdash P$ witnesses that $B$ is not a bar, in that $P$ is an infinite path avoiding $B$. (We take $P$ to be a term for a function such that $P(n)$ is a binary sequence of length $n$.) We want to show that $p$ is an inconsistent node. By the choice of $C$, each $P(n)$ is (forced to be) equal to $\chi(n)$. Let $n$ be a natural number. Then $p \Vdash P(n) \notin B$; that is, $p \Vdash \exists s \geq n$ some extension of $P(n)$ of length $s$ is labeled "out"; that is, $p \Vdash \exists s \geq n\ K_s$ extends $P(n)$. Unpacking the definition of forcing an existential statement, there is a membership function $m$ of $p$, and whenever $m(\rho) = 1$ there is a natural number $s_\rho$ such that $p_\rho \Vdash K_{s_\rho}$ extends $P(n)$.

We will need to leverage the fact that the halting problem has no computable solution. So we now need to work in $\mathcal{M}_{K_1}$, where all of the above is happening. That means there is a realizer $e$ such that $\{e\}(n) \Vdash_r$ "There is a membership function $m$ of $p$, and $\forall \rho\ m(\rho) = 1 \rightarrow \exists s_\rho\ p_\rho \Vdash K_{s_\rho} \supseteq P(n)$." In particular, $m$ is a computable function of $e$ and $n$, as is $h_m$. Go through all tuples $\rho$ of height

$h_m$, and apply $m$ to them. If $m(\rho)$ is never 1, then we have a witness that $p$ is inconsistent, and we are done. If $m(\rho)$ is ever 1, then we can compute $s_\rho$.

If $p$ were consistent, then so is $p_\rho$ for some $\rho$ of height $h_m$. For any such $\rho$, $m(\rho) = 1$ and whatever $p_\rho$ forces is the truth. In particular, $K_{s_\rho}$ really does extend $\chi(n)$. Of course, the converse does not hold: just because $m(\rho) = 1$ does not mean that $p_\rho$ is consistent. But what we could do is compute $s_\rho$ whenever $m(\rho) = 1$ and take their maximum $s$. Then $K_s$ extends $\chi(n)$. If $p$ were consistent, we could do this for every $n$, and would then have a computable solution to the halting problem. Since there is no such thing, $p$ is inconsistent.

We are not quite done yet. We must compute a witness that $p$ is inconsistent. In some detail: What we want is a realizer that $\langle\rangle \Vdash \text{FAN}_c \rightarrow 0 = 1$; that is, a realizer for $\forall p$ if $p \Vdash \text{FAN}_c$ then $p \Vdash 0 = 1$. So we need a computable procedure that would take an $e$ realizing $p \Vdash \text{FAN}_c$ and return a realizer that $p \Vdash 0 = 1$. The only way that $p$ could force 0=1 is with a membership function that never takes on the value 1. It sure helps knowing, as we showed above, that $p$ is inconsistent; what we still must do is compute such a membership function. We now give a procedure that halts with such a function whenever $p$ is inconsistent.

Interleave the following procedures. Because $\mathcal{O}_0^p$ is decidable, go through all binary sequences $\mu$, shorter before longer, and determine which are in $\mathcal{O}_0^p$ and which are not. (Notice that in this base case, the answer in unambiguous: there is no setting in which some $\mu$ is sometimes in and sometimes out of $\mathcal{O}_0^p$.) Whenever $\mu$ is determined not to be in $\mathcal{O}_0^p$, then no extension of $\mu$ need be considered, in either sense of a binary sequence $\nu$ as an end-extension of $\mu$ or of a tuple $\rho$ with $\mu$ (or an end-extension $\nu$) as a $0^{th}$ component. Also, when building a membership function $m$, we will take $m(\rho)$ for any $\rho$ extending $\mu$ to be $\langle 0, 0\rangle$. Inductively, suppose we have $\rho$ of length $k < n$ the length of $p$. We will assume that, for all $i < k$, $p_{\rho\restriction i} \Vdash_H \rho(i) \in \mathcal{O}_i^p$, with the following justification. Because $p \restriction i$ forces $\mathcal{T}_i^p$ to be decidable and $\mathcal{O}_i^p$ as an open set to have a base, by extending the height of the $i$-tuples considered enough, we will eventually be considering $\rho$'s that are big enough to decide membership in $\mathcal{O}_i^p$. If we ever find that $p_{\rho\restriction i} \Vdash_H \rho(i) \notin \mathcal{O}_i^p$ then $\rho$ and any of its extensions will be labeled by any $m$ we build $\langle 0, i\rangle$.

If at any time all $\rho$'s of a fixed height are labeled with some $\langle 0, j\rangle$, then we have a membership function witnessing the inconsistency of $p$. If $p$ really is inconsistent, then we will eventually find such a membership function, as follows. There is a least $i$ such that $p \restriction i \Vdash \mathcal{O}_i^p = \perp$. For that $i$, $p \restriction i \Vdash$ there is a height such that $\mathcal{O}_i^p$ contains nothing of that height. Then there is a height $H$ of length $i$ and integer $G$ such that every $\rho$ of height $H$ is either labeled $\langle 0, j\rangle$ $(j < i)$ or $p_\rho \Vdash \mathcal{O}_i^p$ contains nothing of height $G$; in the latter case, for any $\mu$ of height $G$, $\rho^\frown\mu$ will be labeled $\langle 0, i\rangle$. Notice that the procedure might never get that far. There might well be $i$-tuples $\rho$ and sequences $\mu$ such that $p_\rho \Vdash_H \mu \in \mathcal{O}_i^p$, and so the procedure will continue with $\rho^\frown\mu$, even if $p_\rho \Vdash_H \mathcal{O}_\mu = \perp$, a condition we never checked. Then $p_{\rho^\frown\mu}$ will force anything. In particular, we might find $\rho^\frown\mu$ and its extensions forcing lots of sequences out of $\mathcal{O}_k^p$ for $k > i$, so that we get a desired membership function $m$ (i.e. labeling nothing 1) with some height allowing for $\rho^\frown\mu$. In a sense, this is not really what we want, since it might not be honest: there might be extensions of $p_{\rho^\frown\mu}$ that we consider which force untruths. Nonetheless, this will satisfy the definition of forcing 0=1. $\square$

# References


[1] Michael Beeson, *Foundations of Constructive Mathematics*, Springer-Verlag, 1985.

[2] Josef Berger, *The logical strength of the uniform continuity theorem*, Logical Approaches to Computational Barriers, Proceedings of CiE 2006, LNCS 3988, 2006, pp. 35–39.

[3] ———, *A separation result for varieties of Brouwer's Fan Theorem*, Proceedings of the 10th Asian Logic Conference (ALC 10), Kobe University in Kobe, Hyogo, Japan, September 1-6, 2008, 2010, pp. 85–92.

[4] Hannes Diener and Iris Loeb, *Sequences of real functions on [0, 1] in constructive reverse mathematics*, Annals of Pure and Applied Logic **157(1)** (2009), pp. 50-61.

[5] Hannes Diener and Robert Lubarsky, *Notions of Cauchyness and Metastability*, Symposium on Logical Foundations in Computer Science 2018 (Sergei Artemov and Anil Nerode, eds.), Lecture Notes in Computer Science, Springer Berlin / Heidelberg, 2018.

[6] M. Fourman and J. Hyland, *Sheaf models for analysis*, Applications of Sheaves (Michael Fourman, Christopher Mulvey, and Dana Scott, eds.), Lecture Notes in Mathematics, vol. 753, Springer Berlin / Heidelberg, 1979, pp. 280-301.

[7] M. Fourman and D. Scott, *Sheaves and Logic*, Applications of Sheaves (Michael Fourman, Christopher Mulvey, and Dana Scott, eds.), Lecture Notes in Mathematics, vol. 753, Springer Berlin / Heidelberg, 1979, pp. 302-401.

[8] Harvey Friedman and Andre Scedrov, *The lack of definable witnesses and provably recursive functions in intuitionistic set theories*, Advances in Math (1985), pp. 1-13.

[9] Nicola Gambino, *Heyting-valued interpretations for Constructive Set Theory*, Annals of Pure and Applied Logic **137** (2006), pp. 164-188.

[10] William Julian and Fred Richman, *A uniformly continuous function on [0,1] that is everywhere different from its infimum*, Pacific Journal of Mathematics **111(2)** (1984), pp. 333-340.

[11] Robert Lubarsky, *Intuitionistic L*, Logical Methods in Computer Science: The Nerode Conference (Crossley et al., ed.), Birkhauser, 1993, pp. 555-571.

[12] ———, *Independence Results around Constructive ZF*, Annals of Pure and Applied Logic **132** (2005), pp. 209-225.

[13] ———, *Separating the Fan Theorem and Its Weakenings II*, LNCS 10703: Logical Foundations of Computer Science (Artemov and Nerode, ed.), Springer, 2018, pp. 242-255.

[14] Robert Lubarsky and Hannes Diener, *Separating the Fan Theorem and its weakenings*, Journal of Symbolic Logic **79** (2014), pp. 792-813.

[15] David McCarty, *Realizability and recursive mathematics*, Ph.D. Thesis from Oxford University, and Carnegie-Mellon University Technical Report CMU-CS-84-131, 1984.

[16] ———, *Realizability and recursive set theory*, Annals of Pure and Applied Logic **32** (1986), pp. 153-183.

[17] Jaap van Oosten, *Realizability: An Introduction to its Categorical Side*, Elsevier, 2008.

[18] A.S. Troelstra and D. van Dalen, *Constructivism in Mathematics, Vol. 2*, Studies in Logic, vol. 123, Elsevier, 1988.