Intuitionistic $\mathbb{L}$

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Introduction

The goal of this paper is to develop the basics of $\mathbb{IL}$, that is, L under intuitionistic reasoning. The highlights are that (under IZF) $\mathbb{IL}$ is a model of V=L and also of IZF. While these are not exciting results classically, they and their associated lemmas are examples of the phenomenon that classical trivialities can become sticky intuitionistically, when they're not downright false.

By way of some general background, IZF is ZF set theory with intuitionistic logic in place of classical logic. As is often the case, one must take care what exactly ZF is taken to be, since classically equivalent statements can become inequivalent in a constructive setting. In the case at hand, Foundation, as the axiom that every non-empty set has an $\in$-minimal member, implies Excluded Middle, so is replaced by the schema of $\in$-Induction:

$$\forall y\ [(\forall x \in y\ \ \varphi(x)) \rightarrow \varphi(y)] \rightarrow \forall x\ \varphi(x)\ \ .$$

For a discussion of Infinity, see the section on $I\omega$ below. The only other problematic part of ZF is that, in the presence of the other ZF axioms, Replacement, Collection (aka Bounding), and Reflection are equivalent classically, but the proof of such breaks down intuitionistically. Hence there are three corresponding flavors of intuitionistic ZF: IZF_{Rep}, IZF, and IZF_{Ref}. It is easy to see that IZF_{Ref} implies IZF, which itself implies IZF_{Rep}. That IZF_{Rep} does not imply IZF was first proven in [FS], where it is shown that IZF does not have the existence property, in contrast with IZF_{Ref}. An alternative proof is given in [Li] ~~or [Lu]~~, which contains a Kripke model of $IZF_{Rep} + \neg IZF$. It is unknown whether IZF implies IZF_{Ref}. (These issues also spill over into the Axiom of Separation (aka Comprehension), which is considered classically to be a corollary of Replacement, but intuitionistically is posited separately.)

While any discussion of $\mathbb{IL}$ logically needs some of the sections that follow, it should cause no confusion to mention here that IZF_{Ref} easily proves that $\mathbb{IL}$ models IZF_{Ref}. It is one of our goals to show that under IZF IL models IZF. It is unknown whether IZF_{Rep} proves that IL models the same.

The problem showing $(V=L)^{IL}$ is that the ordinals are not absolute; hence, for $IL = \bigcup_{\alpha\in ORD} IL_\alpha$, it is not clear that $IL = \bigcup_{\alpha\in (ORD)^{IL}} IL_\alpha$. The problem showing that $(IZF)^{IL}$ is not in showing that Collection holds in IL, which is actually trivial. The trouble is Separation (aka Comprehension), which seems to call for some kind of Reflection. The trouble trying to prove Reflection in IZF, which is the same roadblock in trying to prove Collection in IZF_{Rep}, is that the ordinals are not linearly ordered.

Regarding the meta-theory, it would be nice always to work within as weak a theory as possible, to produce the strongest results. For instance, HA should be enough to do all the syntax we need, and the basics of IL should need only IKP, whatever that may turn out to be. However, the question of just what's needed where is a bit removed from our present purposes, and would complicate matters unnecessarily. We will compromise by working within IZF_{Rep}, until the proof of $(IZF)^{IL}$, which clearly needs to be done in IZF.

Of course, to work within IZF_{Rep}, we need IL to be a definable subclass. This depends primarily upon internalizing definability within IZF_{Rep}, which in turn relies upon the coding of syntax. This latter operation is usually done by using the integers to represent symbols, terms, formulas, and such like, in such a manner that the normal syntactic operations become recursive (even primitive recursive) functions. However, it is unclear that the intuitionistic integers can support such a burden. So we begin this paper by developing $I\omega$. Then we examine definability, with an eye toward keeping the witnesses to "Y = def(X)" (i.e. "Y is the collection of definable subsets of X") easily constructible from X. This is then applied to the L-hierarchy itself, and finally the two main theorems are proven.

$\underline{I\omega}$

The Axiom of Infinity is usually taken to be $\exists x\ (\exists y\in x\ \&\ \forall y\in x\, \exists z\in x\ y\in z)$. From this, one uses $\in$-Induction, Power, and Separation to define the least set containing 0 and closed under successor, called (reasonably enough) ω. (See [P] for details.) However, I find it unsatisfying, philosophically and mathematically, to have ω depend on any strong set-theoretic axiom, much less three of them, especially when there is a natural alternative. The Axiom of Infinity could just as well be taken to be:

$$\exists x\ [0\in x\ \&\ \forall y\in x\ \exists z\in x\ (z = y\ \cup \{y\})$$
$$\&\ \forall y\in x\ (y=0 \vee \exists z\in x\ (y = z\ \cup \{z\})\ )\ ].$$

(In what follows, we will use "x+1" as an abbreviation for "$x \cup \{x\}$".) Using $\in$-Induction, it is easy to show that the set posited by the latter Axiom of Infinity is equal to the set constructed by the former. So choose the version of the axiom you prefer, and get on with matters.

Our next needs are that ω models HA (more accurately, that it can be expanded to a model of HA, by suitably defining +, *, etc.), and that HA proves all the facts we'll use about syntax. This is folklore, but since it is not readily accessible in print it might help somebody to include a short discussion of these matters here.

To begin with, let's make clear the strong inductive properties which ω enjoys.

<u>proposition</u> (ω-Induction) For any φ, if $\varphi(0)$ and $\varphi(x) \rightarrow \varphi(x+1)$, then $\forall x\in\omega\ \varphi(x)$.

<u>proof</u> By $\in$-Induction applied to the formula "$x\in\omega \rightarrow \varphi(x)$", it suffices to show:

$$\forall y\ [\ (\forall x\in y\ \ x\in\omega \rightarrow \varphi(x)) \rightarrow (y\in\omega \rightarrow \varphi(y))\ ].$$

So assume $(\forall x\in y\ \ x\in\omega \rightarrow \varphi(x))$ (IH) and $y\in\omega$; we must show $\varphi(y)$. Since $y\in\omega$, either $y=0$ or $\exists z\in\omega\ \ y = z+1$. In the former case $\varphi(y)$, by the hypotheses on φ. In the latter, by IH on z, $\varphi(z)$, and again by the hypotheses on φ $\varphi(z+1)$, i.e. $\varphi(y)$.

<u>proposition</u> (Definition by Primitive Recursion) Suppose $f(x)$ and $g(y, z, x)$ are total functions (where x is an n-tuple of variables ranging over ω^n, and y and z range over ω). There is a unique function h such that:

$$h(0, x) = f(x) \quad \text{and} \quad h(x+1, x) = g(x, h(x, x), x).$$

<u>proof</u> Easy, using ω-Induction.

The standard +, *, and < can now be defined on ω, producing a structure henceforth referred to as $\mathbb{N}$. $\mathbb{N}$ is easily seen to be a model of HA. Some easy corollaries are:

1) = is decidable on $\mathbb{N}$; what's more, < is a linear order.

2) $\mathbb{N}$ supports basic syntax. We will freely talk about formulas, their structure, basic manipulations on them, and such like, and assume that our informal discussion could be formalized within the language of set theory and interpreted correctly within any set (containing $\mathbb{N}$).

3) If φ (in the language of arithmetic) has only bounded quantifiers, then $\mathbb{N} \models \forall x\ \ (\varphi(x) \vee \neg\varphi(x))$. Notice that this corollary could be considered a single theorem of IZF_{Rep}, by 2) above.

Definability

If X is a set, then each subset of X definable over X is also a set (by Separation). In order to claim, though, that the collection of all such subsets of X is a set, we need that the property of being a definable subset of X is itself definable. This can be done in a rather straightforward, hands-on manner. A witness that Y is a definable subset of X is a definition $\varphi(x)$ (Remember that Gödel coding is implicitly assumed wherever necessary.), which may have parameters from X, and a witness that $Y = \{ x \in X \mid X \models \varphi(x) \}$. The latter witness is basically (a superset of) the set of sub-formulas of φ, with parameters, that are true in X; in other words, the witness is an inductively defined truth function on the sub-formulas of φ that tells us that Y is the set in question. The purpose of this section is to spell out this construction in more detail.

It is easy to see that the set of finite functions from the variables in the language to X, called VAR -> X, is a set. For each $n \in \omega$ there will be a truth set Tr_n for formulas of Gödel code $< n$, which will be a subset of $n \times$ (VAR -> X). Tr_n will be a truth set in that $X \models \varphi\,[f]$ iff $<\varphi, f> \in Tr_n$. (Implicity, if $<\varphi, f> \in Tr_n$ then the free variables of φ are in the domain of f.) Clearly, Tr_0 is the empty set. Given Tr_n, Tr_{n+1} depends on the form of the nth formula:

If φ_n = "$x_i \in x_j$",
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid f(x_i) \in f(x_j) \}$.

If φ_n = "$x_i = x_j$",
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid f(x_i) = f(x_j) \}$.

If $\varphi_n = \varphi \,\&\, \psi$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid <\varphi, f> \in Tr_n \,\&\, <\psi, f> \in Tr_n \}$.

If $\varphi_n = \varphi \vee \psi$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid <\varphi, f> \in Tr_n \vee <\psi, f> \in Tr_n \}$.

If $\varphi_n = \varphi \rightarrow \psi$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid <\varphi, f> \in Tr_n \rightarrow <\psi, f> \in Tr_n \}$.

If $\varphi_n = \neg\varphi$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid \neg(<\varphi, f> \in Tr_n) \}$.

If $\varphi_n = \forall x_i\, \varphi(x_i)$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid \forall\, x \in X, <\varphi, f[x_i/x]> \in Tr_n \}$.

If $\varphi_n = \exists x_i\, \varphi(x_i)$,
then $Tr_{n+1} = Tr_n \cup \{ <n, f> \mid \exists\, x \in X, <\varphi, f[x_i/x]> \in Tr_n \}$.

But not only is each Tr_n a set, they are so uniformly definable that the set of Tr_ns is also a set, by replacement. Let $Tr = \bigcup_{n\in\omega} Tr_n$. Then Y is a definable subset of X iff there is a formula $\varphi(x_0, x_1, \ldots, x_k)$ and a function $f : \{x_1, \ldots, x_k\} \to X$ such that $Y = \{x \in X \mid \langle\varphi, f \cup \{\langle x_0, x\rangle\}\rangle \in Tr\}$. Also, Z = def(X) iff $\forall Y$ ($Y \in Z$ iff Y is a definable subset of X) iff (to keep the quantifiers more bounded) $\forall Y \in Z\ \exists\varphi \in \omega\ \exists$ parameters from X (Y is defined over X via φ) & $\forall\varphi \in \omega\ \forall$ parameters from X $\exists Y \in Z$ (Y is defined over X via φ).

II. L

Now that we know what definability is, we can talk intelligently about L.

definition $L_\alpha = \bigcup_{\beta<\alpha} def(L_\beta)$.
$L = \bigcup_{\alpha\in ORD} L_\alpha$.

We must show that each L_α is a set. This hinges upon the uniform definability of $def(L_\alpha)$ as a function of α. The canonical witness that "$X = def(L_\alpha)$" is a pair of canonical witnesses "$Y = L_\alpha$" and "X = def(Y)". "$Y = L_\alpha$" is witnessed canonically by the pair of functions $f(\beta) = def(L_\beta)$ and $g(\beta)$ = the canonical witness to "$f(\beta) = def(L_\beta)$", with domains α, which exist by replacement and induction. "X = def(Y)" is witnessed with Tr as in the previous section. So each L_α exists, and L is a definable sub-class.

Recall that our goal is to show that L is a model of "V=L". This might reasonably be interpreted as meaning that for all $x \in L$ there is an $\alpha \in L$ such that $x \in L_\alpha$. But we can do better than this. We want to be able to do the construction above in L itself, so that "$\forall x\ \exists \alpha \in L\quad x \in L_\alpha$" makes sense in L. In particular, L has to be able to interpret definability, just like V, so at the very least it must be checked that L, which is not known to model IZF_{Rep}, contains the Tr's and the witnesses and so on. This will be done in the next section. Even before that, though, it must be shown that ω is in L. This we turn to now.

The argument below seems a bit circuitous, but even if it is unnecessarily so, it still gives other interesting information about L. Once it is completed, it will be easy to show that the other related things we need are also in L, and rather conveniently located to boot. (Incidentally, if the traditional Axiom of Infinity is accepted instead of the alternate proposed earlier, then the construction of ω is non-absolute. So it would not be enough to show that L models IZF_{Rep}, since L's ω could very well be different from V's.)

The trick here is that we are allowed to use ω to define ω. That is, work over L_ω. The intuition is that bounded assertions about L_ω should be like those for classical L_ω, because we have bounded decidability for ω, so the classical definition of ω over L_ω should do. In order to show that L_ω satisfies classical logic for bounded formulas, we arithmetize it (i.e. code it into the integers), and take advantage of the work already done on $\mathbb{N}$.

Although we will actually work with the decoding function, for the sake of intuition we first define the coding function $\# : L_\omega \to \omega$ inductively. For $X = \{x_0, x_1, \ldots, x_n\}$, $\#X = 2^{\#x_0} + 2^{\#x_1} + \ldots + 2^{\#x_n}$. (The empty set gets the number 0.) The numbers $\#x_i$ are the base two components of $\#X$; the property of being a base two component is the arithmetic counterpart to the membership relation. In what follows we will need a definition for this relation. One is: x is a base two component of y iff [there is a $z < 2^x \leq y$ such that 2^x divides y-z and 2^{x+1} does not divide y-z]. Since "y-z" is not technically a term in the language, say "there is a $q \leq y$ such that $y = z + q$", and refer to q instead.

The decoding function is a 1-1 function f from ω to V such that $f(n) = \{ f(i) \mid i$ is a base two component of n $\}$. Show that for each $n \in \omega$ there is a unique such f_n with domain n, by induction. For n = 0, let f_0 be the empty function. For n+1, first we show uniqueness. $f_{n+1}(n)$ is determined by $f_{n+1} \mid$ (restricted to) n, by the defining property of f_{n+1}. Furthermore, $f_{n+1} \mid n$ satisfies the definition of f_n, which is unique by assumption. Therefore, there could be only one f_{n+1}. To show that there is one, take the union of f_n with the appropriate ordered pair. That this f_{n+1} is 1-1 depends on some numerical facts about ω that are left to the reader. Finally, f is the union of these f_ns. # is f^{-1}.

<u>definition</u> 2_n is a stack of n 2s: $2^0 = 0$, and $2^{n+1} = 2^{2_n}$.

<u>lemma</u> $L_n = f''2_n$, and for $x, y \in L_n$, $x \in y$ iff $\#x$ is a base two component of $\#y$.
<u>proof</u> By induction on n.

For n = 0, check that both sides are the empty set.

For n+1, suppose first that $X \in L_{n+1}$. X is definable over L_n. Using #, this translates to a bounded definition of a subset of 2_n. (The bounds may be beyond 2_n by the translation of membership, but the definition is still bounded in ω.) Now we use this definition (say φ) in $\mathbb{N}$ to construct the integer code for X, as follows. Show inductively through 2_n that there is an $m \in \omega$ which correctly codes initial segments of X: for all $k \leq 2_n$ there is an $m_k < 2^k$

which codes $X \cap k$. The inductive step $k+1$ is where we use decidability to construct m_{k+1}. If $\varphi(k)$ then $m_{k+1} = m_k + 2^k$; if $\neg\varphi(k)$ then $m_{k+1} = m_k$. $\varphi(k) \vee \neg\varphi(k)$, and in either case $m_{k+1} \in \omega$. As X is a subset of L_n, m_{2_n} is our desired code. It is fairly easy to see that the assertion about membership and base two components also carries along.

The other direction is similar. If $m < 2_{n+1}$, clearly it codes a subset of L_n. The definition we would like to give for f(m) is an explicit list of its members: that is, if $f(m) = \{ x_0, x_1, \ldots, x_n\}$, then $f(m) = \{ x \mid x = x_0 \vee \ldots \vee x = x_n \}$. Show that there is such a formula by constructing a sequence correct on initial segments of 2_n, inductively through 2_n. At successor steps, use decidability to show that the result of your decision is indeed a formula. QED

<u>corollary</u> All formulas over L_n are decidable.

<u>corollary</u> $L_\omega = f''\omega$.

This is nice, because we will be able to prove nice things about L_ω by induction on ω, examining only one set at a time, not having to concern ourselves with all possible definitions over L_n, thanks to f.

<u>lemma</u> For all $m, n \in \omega$, if $m < n$ and $f(m), f(n) \in$ ORD, then $f(m) \in f(n)$.
<u>proof</u> Notice that the property of being an ordinal is bounded, hence decidable.
By main induction on m, with subsidiary induction on n.

For $m = 0$, if $n = 0$ then the hypothesis $m<n$ is not met.
For $m = 0$ and $n + 1$, $f(n+1)$ is not empty, since f is 1-1, so it has a member by the decidability of bounded formulas, say x. If $\#(x) = 0$, then we're done. If $\#(x) \neq 0$, then by induction $f(0) \in x$; since ordinals are transitive, $f(0) \in f(n+1)$.

For $m+1$, if $n=0$ then the hypothesis $m<n$ is not met.
For $m+1$ and $n+1$, assume both $f(m+1)$ and $f(n+1)$ are ordinals. By induction on m, each contains as members all those ordinals with codes $\leq m$; moreover, $f(m+1)$ is exactly $\{ x \mid \#(x) \leq m \}$. If $f(n+1)$ has as a member some set x with code between $m+1$ and $n+1$, then by induction $f(m+1) \in x$; since ordinals are transitive, $f(m+1) \in f(n+1)$. If $f(n+1)$ has $f(m+1)$ itself as a member, then we're done again. If neither of those two conditions obtains, then $f(n+1) = f(m+1)$, which contradicts the injectivity of f. QED

lemma There is a largest $m \leq n$ such that f(m) is an ordinal.
proof By induction on n.
For n=0, 0 is as desired.
For n+1, if f(n+1) is an ordinal then n+1 is as desired. If it's not, then apply the inductive hypothesis to n. QED

lemma For all n, if $f(n) \in ORD$ then $f(n) \in \omega$.
proof By induction on n.
f(0) is the empty set, hence $\in \omega$.
If $f(n+1) \in ORD$ then $f(n+1) = \{ f(m) \mid f(m) \in ORD \text{ and } m \leq n \}$. Let k be the largest ordinal $\leq n$, by the previous lemma. It is easy to see that f(n+1) is the ordinal successor of f(k). By induction, $f(k) \in \omega$, so $f(n+1) \in \omega$. QED

lemma n is definable over L_n.
proof By induction on n.
0 is any subset of L_0.
If $n \in def(L_n) = L_{n+1}$, then $n+1 \in def(L_{n+1})$ as $\{ x \mid x \in n \vee x = n \}$.
QED

proposition $\omega \in def(L_\omega)$
proof By the previous lemma, ω is a subset of L_ω. By the penultimate lemma, ORD^{L_ω} is a subset of ω. So $\omega = \{ x \mid x \text{ is an ordinal} \}$, as a definition over L_ω.
QED

Not only is ω definable over L_ω, but so is all the syntax we need too. After all, each bounded part of any syntactic function or relation is a member of L_ω, as can be shown inductively using decidability again, so any desired function or relation can be pieced together as the union of those finite parts that satisfy the right inductive properties on their domains, definably over L_ω.

Proof of $(V=L)^{IL}$

This theorem is an easy corollary of:
Main Lemma For all α there is an $\alpha^* \in L$ such that $L_\alpha = L_{\alpha^*}$.

The proof of the main lemma depends heavily on internalizing the construction of L within L itself. The reader could be excused for thinking that this internalization isn't worth talking about. After all, there must be a half dozen treatments of this classically, and by the very nature of the goal these treatments must be fairly constructive, hence any of them should work intuitionistically. I am tempted in this case to disregard the professional standard against discussing blind alleys in public, since here especially it captures the flavor of the subject. But I will restrain myself merely to pointing out that I tried three different classical proofs, and all of them failed irreparably.

The approach that seems most suited is based on three principles:
1) Be as straightforward as possible. No prenex forms, no universal formulas, no tricks of any kind. In particular, we will re-examine the construction of L already given, which was chosen for its straightforwardness.
2) Realize that you're working inductively. So anything that you do to α you should already have done to β $(\beta \in \alpha)$.
3) Work uniformly. Of course no one would be so foolish as to try to split into cases, but even assuming extra structure on α (for example, proving something only in the case of α a limit) is trouble. Even if you'd be satisfied having something for some ordinals, prove it for all ordinals.

So what do you need to get L_α? Without question you need ω as a parameter, as the carrier of syntax. In conjunction with the principles above, this next definition is immediate.

<u>definition</u> α_{aug} (α augmented) $= \bigcup \{ \beta_{aug} \mid \beta \in \alpha \} \cup \{\alpha\} \cup (\omega+1)$.
α_{aug} is an ordinal, because $\beta \in \beta_{aug}$ (and, inductively, β_{aug} is also an ordinal).

What else do you need? More specifically, what else would you need to get def(L_α) from L_α? One thing is VAR -> X. It's easy to see that there is an $n \in \omega$ such that for all α_{aug} and for all $X \in L_{\alpha_{aug}}$ (such as L_α), VAR -> X $\in L_{\alpha_{aug}+n}$. (To recognize whether something is in VAR -> X all you need is a bit of syntax, i.e. ω, hence the hypothesis of augmentation.) The reason we can't just take n to be 1 is that VAR -> X is a few steps up in V-rank. In fact, n can be chosen to be 4.

Next comes Tr. An examination of the Tr_ns shows that each is simply definable over the previous one, "simply definable" meaning that it needs only one more step in the L-hierarchy. But we can do even better than that: Tr_n in the definition of Tr_{n+1} can be replaced by its own definition (with ω as a parameter). So each Tr_n is in $L_{\alpha_{aug}+7}$, and Tr $\in$ def($L_{\alpha_{aug}+7}$). ("7" means just be generous enough with V-ranks; all we need is some fixed

$n \in \omega$.) Hence, the function "Q = def(X)" is uniformly definable (Δ_1 even) over all β such that $\alpha_{aug} + 7 < \beta$, with parameter $X \in L_{\alpha_{aug}}$. In particular, "Q = def(L_α)" is definable over $L_{\alpha_{aug}+7}$, with parameter L_α, uniformly over α.

The final component involves witnessing everything, which requires taking other witnesses (inductively) and forming ordered pairs, then collecting those pairs into functions, and other simple set-theoretic operations. As above, this needs some fixed finite number of steps. Respecting the principles above (and generalizing from some fixed $n \in \omega$ to an arbitrary γ), the next definition is again immediate.

<u>definition</u> $\alpha +_H \gamma$, hereditary addition, is defined inductively on α:

$$\alpha +_H \gamma = [\bigcup \{\beta +_H \gamma \mid \beta \in \alpha\} \cup \{\alpha\}] + \gamma.$$

"+" refers to (standard) ordinal addition, which can be defined inductively no problem (see [G1] or [G2]). Notice that the first summand is an ordinal, since $\beta < \beta +_H \gamma$ inductively, hence the ordinal sum is well-defined. $(\alpha +_H \gamma)^-$ refers to the first summand above.

<u>proof of main lemma</u> This will be done inductively on α. In order to make the inductive step go through, we actually need a stronger hypothesis. This hypothesis depends on the choice of an $n \in \omega$. n need only be big enough to do all the operations described above. The choice of n will not depend on α; any sufficiently large n will do.

(†) For all α there is an $\alpha^* \in$ def ($L_{(\alpha_{aug} +_H n)^-}$) such that $L_\alpha = L_{\alpha^*}$ and the witness to "X = def(L_{α^*})" is in $L_{\alpha_{aug} +_H n}$, all uniformly.

Assume (†) holds for all $\beta < \alpha$.

Since $\alpha < (\alpha_{aug} +_H n)^-$, $L_\alpha \in L_{(\alpha_{aug} +_H n)^-}$. Let α^* be $\{\beta \mid$ there is a witness to "X = def(L_β)" in $L_{(\alpha_{aug} +_H n)^-}$ and $L_\alpha \supset$ def(L_β) $\}$. Clearly $\alpha^* \in$ def ($L_{(\alpha_{aug} +_H n)^-}$) (using the augmentation when referring to definability) and $L_\alpha \supset L_{\alpha^*}$. Furthermore, using (†) inductively, $L_\alpha = \bigcup_{\beta<\alpha}$ def(L_β) = $\bigcup_{\beta<\alpha}$ def(L_{β^*}), and the witnesses to "X = def(L_{β^*})" are in $L_{\beta_{aug} +_H n}$, hence in $L_{(\alpha_{aug} +_H n)^-}$. So $L_{\alpha^*} = L_\alpha$. Finally, by the choice of n, the witnesses can be built in n steps. That is, for each β in α^* we already have a witness to "X = def(L_β)", so we need a few steps to build the ordered pair <β, <def(L_β), def(L_β)-witness>>, and then one more step to collect them into a function (and then separate the function into the two functions, f(β) = def(L_β) and g(β) = def(L_β)-witness, to be precise). This witnesses "Y = L_α". Simultaneously, we build the witness to "Z = def(L_α)", using L_α as a parameter. Then collect both these. QED

The main lemma can be read as saying that in L there are enough L_αs. It would be nice to know that the function $\alpha \dashrightarrow L_\alpha$ is actually total, too.

lemma Each set of the form $L_{\alpha_{aug} +_H \omega}$ is closed under the L-function (and the associated witnessing functions).

remark By earlier observations about the uniform definability of the function "Y = def(X)", we already know that $L_{\alpha_{aug} +_H \omega}$ is closed under it. Hence in the inductive argument below, we will assume inductively closure under the L- and def- functions, but prove only closure under the L-function.

proof The argument here, by induction on α, is much like that of the preceding proof, with a lot more cases.

$\alpha_{aug} +_H \omega = \bigcup\{ (\alpha_{aug} +_H \omega)^- + n \mid n \in \omega \}$, so we consider all $\gamma \in L_{(\alpha_{aug} +_H \omega)^- + n}$, by induction on n.

n=0: Either $\gamma \in \text{def } L_{\alpha_{aug}}$ or, for some $\beta \in \alpha_{aug}$ and $\delta \in \beta +_H \omega$, $\gamma \in \text{def } L_\delta$. Within the latter possibility, split into cases on β:

CASE I: $\beta \in \eta_{aug}$ for some $\eta \in \alpha$: so $\delta \in \eta_{aug} +_H \omega$, and $\gamma \in L_{\eta_{aug} +_H \omega}$, which is closed under the functions in question, by induction.

CASE II: $\beta = \alpha$: Then $\delta \in (\alpha +_H \omega)^- + n$ for some $n \in \omega$, and this case can be proved by induction on n.

For $n = 0$, we have either $\delta \in \eta +_H \omega$ $(\eta \in \alpha)$ or $\delta = \alpha$. Since $\eta \in \eta_{aug}$, the former possibility was already considered in case I. In the latter possibility, $\gamma \in \text{def } L_\alpha$. Using the inductive hypothesis (or case I), $L_{(\alpha_{aug} +_H \omega)^-}$ contains the witnesses for all the ordinals in L_α; moreover, $L_{(\alpha_{aug} +_H \omega)^-}$ has no trouble recognizing these witnesses as such, since it has ω as a member (which is needed as a parameter). The definition of γ over L_α as $\{ \eta \mid \varphi(\eta) \}$ can then be changed to $\bigcup\{ X \mid X = \text{def } L_\eta \ \&\ \varphi(\eta)^{L_\alpha} \}$ over $L_{(\alpha_{aug} +_H \omega)^-}$.

For n successor the argument is similar to that above. The only observation that need be made is that it is not enough to assume that the various witnesses for $\gamma \in L_{(\alpha +_H \omega)^- + n}$ are all in $L_{\alpha_{aug} +_H \omega}$ (Consider what the induction step would then be.). Rather, we need that for all n there is an n* such that all the witnesses for $\gamma \in L_{(\alpha +_H \omega)^- + n}$ are in $L_{(\alpha_{aug} +_H \omega)^- + n^*}$. The details are left to the reader.

CASE III: $\beta \in \omega+1$ (Notice that this is the lemma for the case $\alpha = 0$.): There is no induction involved here. The argument is an unraveling of the definitions and an analysis of small ordinals, and is left to the reader.

If, on the other hand, $\gamma \in \text{def } L_{\alpha_{aug}}$, notice that by work already done, each $\eta \in L_{\alpha_{aug}}$ has all of its witnesses in $L_{(\alpha_{aug} +_H \omega)^-}$. The argument can then proceed as is case II, n=0 above.

n successor: As in case II, n successor above. QED

theorem In L, the function $\alpha \dashrightarrow L_\alpha$ is total.

proof Sets of the form $L_{\alpha_{aug} +_H \omega}$ are cofinal in L (since L_α is a subset of $L_{\alpha_{aug} +_H \omega}$). Now use the previous lemma. QED

Proof of $(IZF)^{IL}$

Infinity: We have already verified that $\omega \in L$.

Pair and Union: Trivial.

Extensionality: Easy, using that L is a transitive subclass of V.

$\in$-Induction: Just as in the proof of ω-Induction in section on Iω, to show induction for φ in L, use induction in V on the formula "$x \in L \rightarrow \varphi(x)$".

Power: Let $X \in L$. It's true that $\forall Y \in \mathcal{P}(X)\ (Y \in L \rightarrow \exists \alpha\ Y \in L_\alpha)$. By Collection, there is a set B such that $\forall Y \in \mathcal{P}(X)\ (Y \in L \rightarrow \exists \alpha \in B\ Y \in L_\alpha)$. Let β be $TC(B \cap ORD)$. Notice that β is an ordinal; moreover, $L_\beta \supseteq \mathcal{P}^L(X)$. $\mathcal{P}^L(X)$ is then definable over L_β (assuming, WLOG, that $X \in L_\beta$).

Collection: Similar to Power. Let $X \in L$. Assume that $[\forall x \in X\ \exists y\ \varphi(x,y)]^L$. Then $\forall x \in X\ \exists \alpha\ \exists y \in L_\alpha\ [\varphi(x,y)]^L$. By Collection, there is a B such that $\forall x \in X\ \exists \alpha \in B\ \exists y \in L_\alpha\ [\varphi(x,y)]^L$. As above, for $\beta = TC(B \cap ORD)$, L_β suffices.

Separation: This is the hard one. We need to show that

$$[\forall X\ \exists Y\ \ Y = \{x \in X \mid \varphi(x)\}]^L,$$

for all standard formulas φ. This is done by external induction on φ. (That is, pick any φ; do the following construction on its finitely many sub-formulas, in order.) We need a stronger inductive hypothesis, to be able to handle formulas with more than one free variable:

$$[\forall X\ \exists Y\ \ Y = \{ < x_1, \ldots , x_n > \in X^n \mid \varphi(x_1, \ldots , x_n) \}]^L,$$

where the free variables of φ are included among $x_1, \ldots , x_n$ (some of the x_i's may be dummy). The hard cases are $\exists$ and $\forall$.

So suppose φ is $\exists x_0\ \psi(x_0, x_1, \ldots , x_n)$. By Separation in V, let X' be $\{ < x_1, \ldots , x_n > \in X^n \mid \varphi(x_1, \ldots , x_n)^L \}$. Then

$$\forall\ x_1, \ldots , x_n \in X'\ \ \exists\alpha\ \ \exists x_0 \in L_\alpha\ [\psi(x_0, x_1, \ldots , x_n)]^L.$$

By arguments like those for Power and Collection, there is an ordinal β such that

$$\forall\ x_1, \ldots , x_n \in X'\ \ \exists x_0 \in L_\beta\ [\psi(x_0, x_1, \ldots , x_n)]^L.$$

(WLOG $X \in L_\beta$.) Inductively, L contains

$$\{ <x_0, x_1, \ldots , x_n> \in L_\beta^{n+1} \mid [\psi(x_0, x_1, \ldots , x_n)]^L \}.$$

To get the desired Y, project the latter set onto its last n components, and intersect with X^n.

Now let φ be $\forall x_0\ \psi(x_0, x_1, \ldots , x_n)$. Restricting the range of x_0 produces a possibly different subset of X^n, to wit

$$Y_R = \{ < x_1, \ldots , x_n > \in X^n \mid \forall x_0 \in R\ \ \psi(x_0, x_1, \ldots , x_n)^L \}\ (R \in L).$$

Using notation suggestively, refer to the Y we're looking for as Y_L. In V, let $\mathbb{Y}^*$ be $\{ Y \in P(X^n) \mid \exists\ R \in L\ \ Y = Y_R \}$. By Collection, the R's in the definition of $\mathbb{Y}^*$ can be bounded; WLOG the bound can be taken to be L_β. Then $Y_{L_\beta} = Y_L$, as follows. Since $R \supseteq R' \rightarrow Y_{R'} \supseteq Y_R$, $Y_{L_\beta} \supseteq Y_L$. For the converse inclusion, let $< x_1, \ldots , x_n > \in Y_{L_\beta}$, and $x_0 \in L$, say $x_0 \in L_\gamma$. $Y_{L_\gamma} \in \mathbb{Y}^*$, hence $Y_{L_\gamma} = Y_R$ for some $R \in L_\beta$, and $Y_{L_\gamma} \supseteq Y_{L_\beta}$. So $< x_1, \ldots , x_n > \in Y_{L_\gamma}$, and $\psi(x_0, x_1, \ldots , x_n)^L$.

The other cases on φ are the Boolean operations, which can be done locally.

QED

Questions

1. Does L contain all the ordinals?
2. Classically L is thought of as a fattening of the ordinals. More precisely, there is a Δ_1 bijection between L and ORD, uniform over all L_λ, λ limit. Is something like this true for IL? Is L bijectible with ORD^L? Is there a fattening of ORD^L to a larger collection ORD of ordinals which doesn't change L and has L bijectible with ORD?
3. Does IZF_{Rep} prove $(IZF_{Rep})^L$?

4. An understanding of some examples would be nice. For instance, consider the recursive realizability model as developed in McCarty [M]. Is it a model of V=L?

5. What's the relationship between the truth in V of the relativization of a formula to L, and the truth of this formula within L as a structure in its own right? With Tarski or with Kripke models, there is no difference. With realizability there may well be. To be specific, consider the formula "V=L". The first theorem of this paper proves, in IZF, its relativization to L. This means that in McCarty's model, there is a realizer which, when handed a set in L and a reason it's in L (loosely speaking basically an ordinal over which the set is defined), returns a reason in L (i.e. an ordinal in L) it's in L. This does not mean that if we work in L, we will be able to realize "V=L". Such a realizer would have to take a set which just so happens to be in L (but no reason need be given) and return a reason in L that the set is in L. That this is possible is far from clear.

Acknowledgement

I would like to thank Prof. Anil Nerode for bringing the subject of IL to my attention, and for his support while I was doing this work.

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