On the structure of honest elementary degrees

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Abstract. We present some new results, and survey some old results, on the structure of honest elementary degrees. This paper should be a suitable first introduction to the honest elementary degrees.

Introduction

This paper is devoted to the study of the structure of the honest elementary degrees. We present some new results, but this is also a kind of introduction and survey paper. The new material are found in Section 6 and 7. In the remaining sections, we survey the same material as we do in Part I of [**9**], but we give more detailed proofs and more elaborated explanations. This should be the most thorough and readable introduction to the honest elementary degrees available so far. But be aware that we are talking about a technical introduction, and it is beyond the scope of this paper to motivate our study of the honest elementary degrees.

The roots of our subject can be found in subrecursion theory from the 1970s. Some relevant papers are Meyer & Ritchie [**13**] and Machtey [**10, 11, 12**]. The theory of honest elementary degrees, in the form presented here, was developed by Kristiansen in a series of papers (and a thesis) [**4, 5, 7, 8**] ([**6**]) from the 1990s. A considerable number of the results surveyed in Section 2, 3, 4 and 5 was initially published in these papers.

A recent paper by Kristiansen, Schlage-Puchta and Weiermann [**9**] shows how to generalise honest elementary degree theory to so-called honest α-elementary degree theory. This generalisation connects honest degree theory with proof theory and provability of Π^0_2-statements in formal systems for mathematics, e.g. Peano Arithmetic. Such a connection yields a strong motivation for further research in honest degree theory.

1 Preliminaries

We assume the reader is familiar with the most basic concepts of classical computability theory, see e.g. [**14**] or [**16**]. We also assume acquaintance with subrecursion theory and, in particular, with the elementary functions. An introduction to this subject can be found in [**15**] or [**17**]. Here we just state some important basic facts and definitions, see [**15**] and [**17**] for proofs.

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The *initial elementary functions* are the projection functions ($\mathcal{I}_i^n$), the constants 0 and 1, addition ($+$) and modified subtraction ($\dot{-}$). The *elementary definition schemes* are *composition*, that is, $f(\vec{x}) = h(g_1(\vec{x}), \ldots, g_m(\vec{x}))$ and *bounded sum* and *bounded product*, that is, respectively $f(\vec{x}, y) = \sum_{i<y} g(\vec{x}, i)$ and $f(\vec{x}, y) = \prod_{i<y} g(\vec{x}, i)$. A *function is elementary* if it can be generated from the initial elementary functions by the elementary definition schemes. A *relation* $R(\vec{x})$ *is elementary* when there exists an elementary function f with range $\{0, 1\}$ such that $f(\vec{x}) = 0$ iff $R(\vec{x})$ holds. Relations may also be called *predicates*, and we will use the two words interchangeably. A function f has *elementary graph* if the relation $f(\vec{x}) = y$ is elementary. When we can define a function g from the function f plus the initial elementary functions by the elementary schemes, we will say that g *is elementary in* f.

The definition scheme $(\mu z \leq x)[\ldots]$ is called the *bounded μ-operator*, and $(\mu z \leq y)[R(\vec{x}, z)]$ denotes the least $z \leq y$ such that the relation $R(\vec{x}, z)$ holds. Let $(\mu z \leq y)[R(\vec{x}, z)] = 0$ if no such z exists. The elementary functions are closed under the bounded μ-operator. If f is defined by a primitive recursion over g and h and $f(\vec{x}, y) \leq j(\vec{x}, y)$, then f is defined by *bounded primitive recursion* over g, h and j. The elementary functions are closed under bounded primitive recursion, but not under primitive recursion. Moreover, the elementary relations are closed under the operations of the propositional calculus and under bounded quantification, i.e., $(\forall x \leq y)[R(x)]$ and $(\exists x \leq y)[R(x)]$.

Let $2_0^x = x$ and $2_{n+1}^x = 2^{2_n^x}$, and let $\mathcal{S}$ denote the successor function. The class of elementary functions equals the closure of $\{0, \mathcal{S}, \mathcal{I}_i^n, 2^x, \max\}$ under composition and bounded primitive recursion. Given this characterisation of the elementary functions, it is easy to see that for any elementary function f, we have $f(\vec{x}) \leq 2_k^{\max(\vec{x})}$ for some fixed k. It is also easy to see that the class of functions elementary in f is the closure of $\{0, \mathcal{S}, \mathcal{I}_i^n, 2^x, \max, f\}$ under composition and bounded primitive recursion. As remarked above, the elementary functions are not closed under primitive recursion, but the elementary predicates will be closed under (unbounded) primitive recursion, that is, when a predicate $P(\vec{x}, y)$ is defined by $P(\vec{x}, 0) \Leftrightarrow \phi(\vec{x})$ and $P(\vec{x}, y+1) \Leftrightarrow \psi(\vec{x}, P(\vec{x}, y), y)$, then P will be elementary if ϕ and ψ are elementary.

Uniform systems for coding finite sequences of natural numbers are available inside the class of elementary functions. Let $\overline{f}(x)$ be the code number for the sequence $\langle f(0), f(1), \ldots f(x)\rangle$. Then $\overline{f}$ belongs to the elementary functions if f does. We will be quite informal and indicate the use of coding functions with the notations $\langle \ldots \rangle$ and $(x)_i$ where $(\langle x_0, \ldots, x_i, \ldots, x_n\rangle)_i = x_i$. (So $(x, i) \mapsto (x)_i$ is an elementary function.) Our coding system is monotone, i.e., $\langle x_0, \ldots, x_n\rangle < \langle x_0, \ldots, x_n, y\rangle$ holds for any y, and $\langle x_0, \ldots, x_i, \ldots, x_n\rangle < \langle x_0, \ldots, x_i+1, \ldots, x_n\rangle$. All the closure properties of the elementary functions can be proved by using Gödel numbering and coding techniques.

For unary functions f, g, we use $f \leq g$ to denote $\forall x \in \mathbb{N}[f(x) \leq g(x)]$, and we use f^k to denote the k^{th} iterate of the function f, that is, $f^0(x) = x$ and $f^{k+1}(x) = f f^k(x)$.

2 The honest elementary degrees and the growth theorem

Definition 2.1 *A function $f : \mathbb{N} \to \mathbb{N}$ is* honest *if it is monotone ($f(x) \leq f(x+1)$), dominates 2^x ($f(x) \geq 2^x$) and has elementary graph.* □

Note that when f is honest, we have $f^{y+1}(x) > f^y(x)$, but we do not necessarily have $f(x+y) > f(x)$. From now on, we reserve the letters $f, g, h, \ldots$ to denote honest functions. Small Greek letter like $\phi, \psi, \xi, \ldots$ will denote number-theoretic functions not necessarily being honest.

Definition 2.2 *A function ϕ is* elementary in *a function ψ, written $\phi \leq_E \psi$, if ϕ can be generated from the initial functions ψ, 2^x,* max*,* 0*, $\mathcal{S}$ (successor), $\mathcal{I}_i^n$ (projections) by composition and bounded primitive recursion.*

We define the relation $\equiv_E$ by $f \equiv_E g \Leftrightarrow f \leq_E g \wedge g \leq_E f$. Now, $\equiv_E$ is an equivalence relation on the honest functions, and we will use $\mathcal{H}$ denote the set of $\equiv_E$-equivalence classes of honest functions. The elements of $\mathcal{H}$ are the honest elementary degrees. *Honest elementary degrees will normally just be called* degrees*, and following the tradition of classical computability theory, we use boldface lowercase Latin letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \ldots$ to denote our degrees.*

We will use deg(f) denote the degree of the honest function f, that is, deg(f) $= \{g \mid g \equiv_E f\}$.

We define the relation $<_E$ by $f <_E g \Leftrightarrow f \leq_E g \wedge g \not\leq_E f$; and the relation $|_E$ by $f \,|_E\, g \Leftrightarrow f \not\leq_E g \wedge g \not\leq_E f$. We will use $<, \leq, |$ to denote the relations induced on the degrees by respectively $<_E, \leq_E, |_E$. We use standard, and presumably very familiar, language with respect to these ordering relations, and we will, e.g., say that f lies below *g if $f \leq_E g$; that g* is strictly above *f if $f <_E g$; that $\mathbf{c}$ lies* strictly between *$\mathbf{a}$ and $\mathbf{b}$ if $\mathbf{a} < \mathbf{c} < \mathbf{b}$; that $\mathbf{a}$ and $\mathbf{b}$ are* incomparable *if $\mathbf{a} \mid \mathbf{b}$; and so on.* □

Theorem 2.3 (Growth Theorem) *Let f and g be honest functions. Then, we have*

$$g \leq_E f \quad \Leftrightarrow \quad g \leq f^k \text{ for some fixed } k \; .$$

Proof. Recall that f is monotone and dominates 2^x. By induction on the build-up of a function ψ form the initial functions $0, \mathcal{S}, \mathcal{I}_i^n, 2^x, \max, f$ by composition and bounded primitive recursion, it is easy to prove that there exists $k \in \mathbb{N}$ such that $\psi(\vec{x}) \leq f^k(\max(\vec{x}))$. Hence, if $g \leq_E f$, we have $g \leq f^k$ for some fixed k.

Now, suppose that $g \leq f^k$. Since g is honest, the relation $g(x) = y$ is elementary. We have $g(x) = (\mu y \leq f^k(x))[g(x) = y]$. Hence, $g \leq_E f$ since the functions elementary in f are closed under composition and the bounded μ-operator. □

The structure of honest elementary degrees is comparable to a classical computability-theoretic degree structure, e.g., the structure of Turing degrees, but the Growth Theorem makes it possible to abandon classical computability-theoretic proof methods and investigate this structure by asymptotic analysis and methods of number theoretic nature. To prove that $g \leq_E f$, it is sufficient to provide a fixed k such that $g(x) \leq f^k(x)$; to prove that $g \not\leq_E f$, it is sufficient to prove that such a k does not exist. Thus, there is no need[1] for the standard computability-theoretic machinery involving enumerations, diagonalisations and constructions with requirements to be satisfied. This makes the proofs concise and transparent.

3 The lattice of honest elementary degrees

Definition 3.1 Least upper bounds *and* greatest lower bounds *are defined the usual way, and a partially ordered structure where each pair of elements has both a least upper bound and a greatest lower bound is called* a lattice.

[1]Well, at least we can achieve a lot without resorting to such a machinery. See Section 6.

We define the join *of the honest functions* f *and* g*, written* $\max[f,g]$*, by*

$$\max[f,g](x) \;=\; \max(f(x),g(x)) \;.$$

We define the meet *of the honest functions* f *and* g*, written* $\min[f,g]$*, by*

$$\min[f,g](x) \;=\; \min(f(x),g(x)) \;.$$

□

Lemma 3.2 *Let* f *and* g *be honest functions. Then,* $\max[f,g]$ *and* $\min[f,g]$ *are honest functions.*

Proof. It is trivial that $\max[f,g]$ and $\min[f,g]$ are monotone and dominate 2^x. To verify that $\max[f,g]$ and $\min[f,g]$ have elementary graphs, observe that $\max[f,g](x) = y$ holds iff

$$(f(x) = y \wedge (\exists i \leq y)[g(x) = i]) \;\vee\; (g(x) = y \wedge (\exists i < y)[f(x) = i])$$

and that $\min[f,g](x) = y$ holds iff

$$(f(x) = y \wedge (\forall i \leq y)[g(x) \neq i]) \;\vee\; (g(x) = y \wedge (\forall i < y)[f(x) \neq i]) \;.$$

The relations $f(x) = y$ and $g(x) = y$ are elementary. Furthermore, the elementary relations are closed under bounded quantification and the operations of the propositional calculus. Hence, both $\max[f,g](x) = y$ and $\min[f,g](x) = y$ are elementary relations. □

Lemma 3.3 *Let* f *and* g *be honest functions. Then, we have*

$$\min(f^m(x), g^n(x)) \leq \min[f,g]^{m+n}(x) \;.$$

Proof. We prove this lemma by induction on $m+n$. The lemma holds trivially when $m = 0$ or $n = 0$. Now, assume that $m > 0$ and $n > 0$. Then, w.l.o.g. we may assume that $\min[f,g](x) = f(x)$. Together with the induction hypothesis this yields

$$\min(f^m(x), g^n(x)) \;\leq\; \min(f^{m-1}(f(x)), g^n(f(x))) \;\leq\; \min[f,g]^{m-1+n}(f(x)) \;=\; \min[f,g]^{m+n}(x) \;.$$

□

Lemma 3.4 *Let* f, g, h *be honest functions. (i)* $\min[f,g] \leq_E f$ *and* $\min[f,g] \leq_E g$*. (ii) If* $h \leq_E f$ *and* $h \leq_E g$*, then* $h \leq_E \min[f,g]$*.*

Proof. We prove (ii). Assume $h \leq_E f$ and $h \leq_E g$. By the Growth Theorem we have m, n such that $h(x) \leq f^m(x)$ and $h(x) \leq g^n(x)$. By Lemma 3.3, we have

$$h(x) \;\leq\; \min(f^m(x), g^n(x)) \leq\; \min[f,g]^{n+m}(x) \;.$$

By another application of the Growth Theorem, we have $h \leq_E \min[f,g]$. This proves (ii). The proof of (ii) is straightforward by the Growth Theorem. □

Lemma 3.5 *Let* f, g, h *be honest functions. (i)* $f \leq_E \max[f,g]$ *and* $g \leq_E \max[f,g]$*. (ii) If* $f \leq_E h$ *and* $g \leq_E h$*, then* $\max[f,g] \leq_E h$*.*

Proof. Both (i) and (ii) follow straightforwardly from the Growth Theorem. □

Lemma 3.6 *For any honest functions* f, f_1, g, g_1 *such that* $f \leq_E f_1$ *and* $g \leq_E g_1$*, we have (i)* $\min[f,g] \leq_E \min[f_1,g_1]$ *and (ii)* $\max[f,g] \leq_E \max[f_1,g_1]$*.*

Proof. Now, $\leq_E$ is transitive, and thus, (i) follows immediately from Lemma 3.4, and (ii) follows immediately from Lemma 3.5. □

Our previous lemma entails that

$$(f \equiv_E f_1 \ \wedge \ g \equiv_E g_1) \ \Rightarrow \ (\max[f,g] \equiv_E \max[f_1,g_1] \ \wedge \ \min[f,g] \equiv_\alpha \min[f_1,g_1])$$

when f, f_1, g, g_1 are honest functions. By Lemma 3.2, we know that $\max[f,g]$ and $\min[f,g]$ are honest functions whenever f and g are. Hence, the next definition makes sense.

Definition 3.7 *Let f and g be honest functions such that* $\deg(f) = \mathbf{a}$ *and* $\deg(g) = \mathbf{b}$. *We define the* join of $\mathbf{a}$ and $\mathbf{b}$, *written* $\mathbf{a} \cup \mathbf{b}$, *by* $\mathbf{a} \cup \mathbf{b} = deg(\max[f,g])$. *We define the* meet of $\mathbf{a}$ and $\mathbf{b}$, *written* $\mathbf{a} \cap \mathbf{b}$, *by* $\mathbf{a} \cap \mathbf{b} = deg(\min[f,g])$. □

Theorem 3.8 (Distributive Lattice) *The structure $\langle \mathcal{H}, \leq, \cup, \cap \rangle$ is a distributive lattice, that is, for any $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{H}$, we have (i) $\mathbf{a} \cap \mathbf{b}$ is the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$ under the ordering $\leq$; (ii) $\mathbf{a} \cup \mathbf{b}$ is the least upper bound of $\mathbf{a}$ and $\mathbf{b}$ under the ordering $\leq$; (iii) $\mathbf{a} \cup (\mathbf{b} \cap \mathbf{c}) = (\mathbf{a} \cup \mathbf{b}) \cap (\mathbf{a} \cup \mathbf{c})$ and $\mathbf{a} \cap (\mathbf{b} \cup \mathbf{c}) = (\mathbf{a} \cap \mathbf{b}) \cup (\mathbf{a} \cap \mathbf{c})$.*

Proof. It follows from Lemma 3.4 (i) that $\mathbf{a} \cap \mathbf{b}$ is a lower bound of $\mathbf{a}$ and $\mathbf{b}$, and by Lemma 3.4 (ii), $\mathbf{a} \cap \mathbf{b}$ is indeed the greatest lower bound of $\mathbf{a}$ and $\mathbf{b}$. This proves (i). The proof of (ii) is symmetric: use Lemma 3.5 in place of Lemma 3.4. Finally, (iii) holds since $\max(x, \min(y,z)) = \min(\max(x,y), \max(x,z))$ and $\min(x, \max(y,z)) = \max(\min(x,y), \min(x,z))$. □

Let $\mathbf{a}$ and $\mathbf{b}$ be two degrees such that $\mathbf{a} \leq \mathbf{b}$. Now, we do not necessarily have $f \leq g$ for any $f \in \mathbf{a}$ and $g \in \mathbf{b}$. But there will always be some $f \in \mathbf{a}$ and some $g \in \mathbf{b}$ such that we have $f(x) \leq g(x)$, or even $f(x) < g(x)$, for all x. This is consequence of the lemmas above: Pick an arbitrary $f_1 \in \mathbf{a}$ and an arbitrary $g_1 \in \mathbf{b}$, and let $f = \min[f_1, g_1]$ and $g = \max[f_1, g_1]$. Now we obviously have and $f(x) \leq g(x) < g^2(x)$ for all x, but we also have $f \in \mathbf{a}$ and $g, g^2 \in \mathbf{b}$.

Theorem 3.9 (Density-Splitting) *Let $\mathbf{a}$ and $\mathbf{b}$ be degrees such that $\mathbf{a} < \mathbf{b}$. Then, there exist incomparable degrees $\mathbf{c}_0$ and $\mathbf{c}_1$ such that $\mathbf{a} = \mathbf{c}_0 \cap \mathbf{c}_1$ and $\mathbf{b} = \mathbf{c}_0 \cup \mathbf{c}_1$.*

Proof. Pick honest functions f and g such that $\deg(g) = \mathbf{a} < \mathbf{b} = \deg(f)$ and $g(x) < f(x)$. We define the sequence $d_0 < d_1 < d_2 < \ldots$. Let $d_0 = 0$, let d_{2i+1} be the least y such that

$$(\dagger) \qquad (\exists z \leq y) \, [\, f(z) \leq y \ \wedge \ (\exists w \leq z) \, [\, d_{2i} \leq w \ \wedge \ g^i(w) < f(w) \,] \,]$$

and let $d_{2i+2} = f(d_{2i+1})$. Next we define the functions h_0 and h_1. For $\jmath \in \{0,1\}$ let $h_\jmath(x) = \max(H_\jmath(x), g(x))$ where

$$H_\jmath(x) = \begin{cases} f(x) & \text{if } d_{4i+2\jmath} \leq x \leq d_{4i+2\jmath+1} \text{ for some } i \\ H_\jmath(x-1) & \text{otherwise} \end{cases}$$

Since $f \not\leq_E g$, there will for each i exist infinitely many z such that $g^i(z) < f(z)$. Thus, there will always be a number satisfying the definition of d_{2i+1}, and thus the sequence $d_0 < d_1 < d_2 < \ldots$ is well defined.

We will now prove that h_1 and h_2 are honest functions. First, we will argue that the relation $d_i = y$ is elementary. This is not obvious as a relation like $g^i(w) < f(w)$ is not necessarily elementary even if f and g are honest functions. However, the relation $g^i(w) < f(w) \leq y$ will be elementary (in i, w and y) whenever g and f are honest. Now, the statement $(\dagger)$ involved in the definition of $d_i = y$ is equivalent to

$$(\exists z \leq y) \, [\, f(z) \leq y \ \wedge \ (\exists w \leq z) \, [\, d_{2i} \leq w \ \wedge \ g^i(w) < f(w) \leq y \,] \,] \, .$$

Moreover, the elementary relations are closed under primitive recursion, bounded quantifiers and propositional operations. Thus, $d_i = y$ is indeed an elementary relation. When we know that $d_i = y$ is elementary, it becomes easy to see that h_0 and h_1 have elementary graphs. Furthermore, it is obvious that h_0 and h_1 are monotone and dominate 2^x, and thus, we are dealing with two honest functions.

Next, we will prove that $\min[h_0, h_1] \equiv_E g$, that $\max[h_0, h_1] \equiv_E f$, and that $h_0 \mid_E h_1$. The theorem follows.

We start by proving $\min[h_0, h_1] \equiv_E g$. By the Growth Theorem it suffices to prove that $\min[h_0, h_1](x) = g(x)$. Assume we have $d_{4i+2} \leq x < d_{4i+4}$. Then

$$\begin{array}{rcll} h_0(x) & = & \max(H_0(x), g(x)) & \text{def. of } h_0 \\ & = & \max(H_0(d_{4i+1}), g(x)) & \text{def. of } H_0 \\ & = & \max(f(d_{4i+1}), g(x)) & \text{def. of } H_0 \\ & = & \max(d_{4i+2}, g(x)) & \text{def. of } d_{4i+2} \\ & = & \max(x, g(x)) & \text{as } d_{4i+2} \leq x \\ & = & g(x) & \text{as } g(x) \geq 2^x \end{array}$$

A symmetric argument shows that $h_1(x) = g(x)$ when there exists i such that $d_{4i} \leq x < d_{4i+2}$. Hence, for any x, we either have $h_0(x) = g(x)$ or $h_1(x) = g(x)$, and since $\min[h_0, h_1](x) \geq g(x)$, we can conclude that $\min[h_0, h_1](x) = g(x)$. This proves that $\min[h_0, h_1] \equiv_E g$.

Our next task is to prove that $\max[h_0, h_1] \equiv_E f$. It follows straightaway from our definitions that we have $\max[h_0, h_1](x) \leq f(x)$. We will prove that $f(x) \leq \max[h_0, h_1]^2(x)$, and thus, we have $\max[h_0, h_1] \equiv_E f$ by the Growth Theorem. The proof of $f(x) \leq \max[h_0, h_1]^2(x)$ splits into two cases. *Case (i):* When x is in the interval $d_{2i} \ldots d_{2i+1} - 1$ for some i, we have $f(x) \leq \max[h_0, h_1]^2(x)$ as either h_0 or h_1 will equal f in this interval. *Case (ii):* Assume x is in the interval $d_{2i+1} \ldots d_{2i+2} - 1$ for some i, and note that

$$h_0(d_j) = f(d_j) \text{ or } h_1(d_j) = f(d_j) \qquad (*)$$

holds for any j. We have

$$\begin{array}{rcll} f(x) & \leq & f(d_{2i+2}) & f \text{ is monotone} \\ & = & \max[h_0, h_1](d_{2i+2}) & (*) \\ & = & \max[h_0, h_1](f(d_{2i+1})) & \text{def. of } d_{2i+2} \\ & = & \max[h_0, h_1]^2(d_{2i+1}) & (*) \\ & \leq & \max[h_0, h_1]^2(x) & \max[h_0, h_1] \text{ is monotone} \end{array}$$

This completes the proof of $\max[h_0, h_1] \equiv_E f$.

Finally, we prove $h_0 \mid_E h_1$. Fix an arbitrary $m \in \mathbb{N}$. We will argue that there exists x such that $h_0^m(x) < h_1(x)$. Let $k \geq 2m$. By the definition of d_{4k+3} there exists a number x_k in the interval $d_{4k+2}, \ldots, d_{4k+3}$ such that

$$d_{4k+2} \ \leq \ g^m(x_k) \ \leq \ g^k(x_k) \ < \ f(x_k) \ \leq \ d_{4k+3} \ . \qquad (\dagger)$$

Now, since $d_{4k+2} \leq x_k \leq g^k(x_k) < d_{4k+3}$, it follows from the definitions of h_0 and H_0 that

$$(\ddagger) \quad h_0(g^\ell(x_k)) = \max(H_0(g^\ell(x_k)), gg^\ell(x_k)) = \max(H_0(d_{4k+1}), g^{\ell+1}(x_k)) = \max(d_{4k+2}, g^{\ell+1}(x_k)) = \max(x_k, g^{\ell+1}(x_k)) = g^{\ell+1}(x_k)$$

holds for any $\ell < k$. When we combine (†), (‡) and the definition of h_1, we get $h_0^m(x_k) = g^m(x_k) \leq g^k(x_k) < f(x_k) = h_1(x_k)$. This proves that, for any m, we can find x such that $h_0^m(x) < h_1(x)$. By the Growth Theorem, we have $h_1 \not\leq_E h_0$. The proof that $h_0 \not\leq_E h_1$ is symmetric. Hence, $h_0 \mid_E h_1$. □

Results being obviously equivalent to Theorem 3.8 and Theorem 3.9, are proved by Machtey [**11, 12**] by traditional computability-theoretic methods.

4 A jump operator on honest elementary degrees

We will now define an operator $\cdot'$ transforming an honest function f into a faster increasing honest function f'. This operator will be called the *jump operator*.

Definition 4.1 *For any honest function f, we define* the jump of f, *written f', by $f'(x) = f^{x+1}(x)$.* □

Lemma 4.2 *Let f be an honest function. Then, f' is an honest function.*

Proof. It is obvious that f' is monotone and dominates 2^x. Let $\psi(x, y)$ be an elementary function that places a bound on the code number for the sequence $\langle y, y, \ldots, y\rangle$ of length $x + 1$. Then, $f'(x) = y$ is equivalent to

$$(\exists s \leq \psi(x, y))[(s)_0 = f(x) \ \wedge \ (\forall i < x)[(s)_{i+1} = f((s)_i)] \ \wedge \ (s)_x = y] .$$

Thus, the relation $f'(x) = y$ is elementary since all the functions, relations and operations involved in this expression are elementary. This proves that f' has elementary graph. □

Lemma 4.3 (Monotonicity of the Jump Operator) *Let f and g be honest functions. Then, we have*

$$g \leq_E f \quad \Rightarrow \quad g' \leq_E f' .$$

Proof. Suppose $g \leq_E f$. By the Growth Theorem, we have a fixed k such that $g(x) \leq f^k(x)$. Now

$$g'(x) = g^{x+1}(x) \leq (f^k)^{x+1}(x) \leq f^{(kx+k)+1}(kx + k) = f'(kx + k) \leq (f')^{2k}(x)$$

and $g' \leq_E f'$ follows by another application of the Growth Theorem. □

Lemma 4.3 entails that $f' \equiv_E g'$ whenever f and g are honest functions such that $f \equiv_E g$. Hence, the jump operator on the honest functions induce an operator on the honest elementary degrees.

Definition 4.4 *For any honest elementary degree* $\mathbf{a}$, *we define* the jump of $\mathbf{a}$, *written $\mathbf{a}'$, by $\mathbf{a}' = deg(f')$ where f is some honest function such that $\mathbf{a} = deg(f)$. Furthermore, we define the* zero degree, *written $\mathbf{0}$, by $\mathbf{0} = deg(2^x)$.* □

The proof of the next theorem is straightforward. See, Kristiansen [**5**] for the details.

Theorem 4.5 (Canonical Degrees) *We have $\mathbf{0} < \mathbf{0}' < \mathbf{0}'' < \ldots$. Furthermore, $\mathbf{0}$ is the least degree, that is, $\mathbf{0} \leq \mathbf{a}$ holds for any degree $\mathbf{a}$.*

The jump operators of classical computability theory are defined by enumerating all the functions reducible to an oracle function f, e.g., the Turing jump $\mathcal{J}(f)$ of the function f is defined by $\mathcal{J}(f)(\langle e,x\rangle) = \{e\}^f(x)$ where $\{e\}^f$ denotes the e^{th} function Turing computable in f and $\langle\cdot,\cdot\rangle$ is a computable bijection from $\mathbb{N}\times\mathbb{N}$ into $\mathbb{N}$. Jump operators based on enumerations are considered to be natural. The reader should note that our jump operator is equivalent to such a natural jump operator of classical computability theory: Let $\{[i]^f\}_{i\in\mathbb{N}}$ be an elementary enumeration of the functions elementary in the honest functions f, and let $\mathcal{J}(f)(\langle e,x\rangle) = [e]^f(x)$ where $\langle\cdot,\cdot\rangle$ is an elementary bijection from $\mathbb{N}\times\mathbb{N}$ into $\mathbb{N}$. Then, we indeed have $f' \equiv_E \mathcal{J}(f)$. For a proof and further details, see [**5**] and [**6**].

However, in our context, the advantage of defining f' as an iteration of f is obvious: The Growth Theorem is very well suited for dealing with a jump operator based on iterations; we can introduce the canonical degrees $\mathbf{0},\mathbf{0}',\ldots$, and proceed the development of our degree theory, without resorting to enumerations and the apparatus of classical computability theory.

Definition 4.6 *We define the* n^{th} jump of an honest degree $\mathbf{a}$ (function f)*, written* $\mathbf{a}^{[n]}$ *(*$f^{[n]}$*), by* $\mathbf{a}^{[0]} = \mathbf{a}$ *and* $\mathbf{a}^{[n+1]} = \mathbf{a}^{[n]\prime}$ *(*$f^{[0]} = f$ *and* $f^{[n+1]} = f^{[n]\prime}$*). A degree* $\mathbf{a}$ *strictly below* $\mathbf{0}'$ *is* low$_n$ *if* $\mathbf{a}^{[n]} = \mathbf{0}^{[n]}$*, and* high$_n$ *if* $\mathbf{a}^{[n]} = \mathbf{0}^{[n+1]}$. □

Our strategy for proving the existence of low$_n$ and high$_n$ degrees, will be as follows: First we provide degrees $\mathbf{a}_\ell$ and $\mathbf{a}_h$ strictly between $\mathbf{0}^{[n]}$ and $\mathbf{0}^{[n+1]}$ such that $\mathbf{a}'_\ell = \mathbf{0}^{[n+1]}$ $\mathbf{a}'_h = \mathbf{0}^{[n+2]}$. Thereafter we prove that for any degree $\mathbf{b}$ strictly between $\mathbf{0}^{[k+1]}$ and $\mathbf{0}^{[k+2]}$, we can find a degree $\mathbf{c}$ strictly between $\mathbf{0}^{[k]}$ and $\mathbf{0}^{[k+1]}$ such that $\mathbf{c}' = \mathbf{b}$.

Theorem 4.7 *Let f be a strictly monotone and honest function. Then, there exists an honest function g such that $f <_E g$ and $g' \equiv_E f'$.*

Proof. Let $g(x) = f'f(f')^{-1}(x)$ where $(f')^{-1}$ denotes the inverse of f' given by

$$(f')^{-1}(x) = (\mu i)[f'(i) \geq x] \ .$$

Since f' is strictly monotone, we have $(f')^{-1}f'(x) = x$ and $f'(f')^{-1}(x) \geq x$. Furthermore, we have $g(x) = y$ iff

$$(\exists u, v < y)\,[\,(\forall w < u)[f'(w) < x] \ \wedge\ f'(u) \geq x \ \wedge\ f(u) = v \ \wedge\ f'(v) = y\,]$$

and thus, it is easy to see that the graph of g is elementary. It is also easy to see that g is monotone and dominates 2^x. Hence, g is an honest function.

Now, $f(x) \leq ff'(f')^{-1}(x) \leq f'f(f')^{-1}(x) = g(x)$, and for any fixed k and sufficiently large x, we have

$$\begin{array}{rcll}
f^k(x) & \leq & f^kf'(f')^{-1}(x) & \\
 & = & f^kf^{(f')^{-1}(x)+1}((f')^{-1}(x)) & \text{def. of } f' \\
 & \leq & f^{k+(f')^{-1}(x)+1}(k+(f')^{-1}(x)) & \\
 & = & f'(k+(f')^{-1}(x)) & \text{def. of } f' \\
 & < & f'(f(f')^{-1}(x)) & f(x) \geq 2^x \text{ and } x \text{ is large} \\
 & = & g(x)\ . & \text{def. of } g
\end{array}$$

Hence, we have $f <_E g$ by the Growth Theorem.

Next, we observe that $g^k(x) = f'g^k(f')^{-1}(x)$ for any $k > 0$. This is trivially true when $k = 1$, and, by an induction hypothesis, we have

$$g^{k+1}(x) \;=\; gg^k(x) \;=\; gf'f^k(f')^{-1}(x) \;=\; f'f(f')^{-1}f'f^k(f')^{-1}(x) \;=\; f'f^{k+1}(f')^{-1}(x)\ .$$

Thereby, $g'(x) = g^{x+1}(x) = f'f^{x+1}(f')^{-1}(x) \le f'f^{x+1}(x) = f'f'(x)$, and then we have $g' \le_E f'$ by the Growth Theorem. Since $f <_E g$, we also have $g' \equiv_E f'$ by the monotonicity of the jump operator. □

Theorem 4.8 *Let f be an honest function. Then, there exists an honest function g such that $g <_E f'$ and $g' \equiv_E f''$.*

Proof. For any $i \in \mathbb{N}$, let $d_{3i+1} = f''(d_{3i})$, let $d_{3i+2} = f'(d_{3i+1})$, and let $d_{3i+3} = f'(d_{3i+2})$. Let $d_0 = 0$. Furthermore, let

$$G(x) = \begin{cases} f'(x) & \text{if } d_{3i} \le x \le d_{3i+1} \text{ for some } i \\ G(x-1) & \text{otherwise} \end{cases}$$

and let $g(x) = \max(G(x), f(x))$. It is easy to check that g is honest.

First we prove that $f'' \equiv_E g'$. Observe that for any $j \le d_{3i+1} + 1$, we have $d_{3i} \le (f')^j(d_{3i}) \le (f')^{d_{3i}+1}(d_{3i}) = f''(d_{3i}) = d_{3i+1}$. Hence, by the definition of g, we have

$$(*) \qquad f''(d_{3i}) = (f')^{d_{3i}+1}(d_{3i}) = g^{d_{3i}+1}(d_{3i}) = g'(d_{3i})$$

for any $i \in \mathbb{N}$. Now, let x be arbitrary and let i be the unique number such that $d_{3i} \le x < d_{3i+3}$. Then

$$\begin{array}{rcll}
(g')^4(x) & \ge & (g')^4(d_{3i}) & \text{as } g' \text{ is monotone} \\
& = & (g')^3 f''(d_{3i}) & (*) \\
& = & (g')^3(d_{3i+1}) & \text{def. of } d_{3i+1} \\
& \ge & (g')(f')^2(d_{3i+1}) & \text{as } f(x) \le g(x) \\
& \ge & (g')(d_{3i+3}) & \text{def. of } d_{3i+3} \\
& = & f''(d_{3i+3}) & (*) \\
& \ge & f''(x)\ . & \text{as } f'' \text{ is monotone}
\end{array}$$

This proves $f'' \le (g')^4$, and $f'' \le_E g'$ follows by the Growth Theorem. Moreover, since $g \le f'$, we have $g \le_E f'$, and thus also $g' \le_E f''$ by the monotonicity of the jump operator. This proves that $f'' \equiv_E g'$.

Next we prove that $g <_E f'$. It is obvious that $g \le_E f'$ since $g(x) \le f'(x)$. Hence, we are left to prove that $f' \not\le_E g$. Assume $d_{3i+2} \le x < d_{3i+3}$. Then, straightaway form the definition of g and the sequence $\{d_j\}_{j\in\mathbb{N}}$, we have

$$g(x) \;=\; \max(G(x), f(x)) \;=\; \max(G(d_{3i+1}), f(x)) \;=\; \max(f'(d_{3i+1}), f(x)) \;=\; \max(d_{3i+2}, f(x)) \;=\; \max(x, f(x)) \;=\; f(x)$$

that is, $g(x) = f(x)$ holds for any x in the interval $d_{3i+2}, \ldots, d_{3i+3} - 1$. Let k be an arbitrary fixed number, and pick any i such that $d_{3i+2} + 1 > k$. Then,

$$d_{3i+3} \;=\; f'(d_{3i+2}) \;=\; f^{d_{3i+2}+1}(d_{3i+2}) \;>\; f^k(d_{3i+2}) \;=\; g^k(d_{3i+2})\ .$$

The last equality holds since we have $d_{3i+2} \le f^{\ell}(d_{3i+2}) < d_{3i+3}$ when $\ell \le k$. This, proves that for any fixed k there exists x such that $f'(x) > g^k(x)$, and thus the the Growth Theorem yields $f' \not\le_E g$. $\Box$

Corollary 4.9 *For any n, there exists degrees $\mathbf{a}_\ell$ and $\mathbf{a}_h$ strictly between $\mathbf{0}^{[n]}$ and $\mathbf{0}^{[n+1]}$ such that $\mathbf{a}'_\ell = \mathbf{0}^{[n+1]}$ and $\mathbf{a}'_h = \mathbf{0}^{[n+2]}$.*

Proof. Let f be an honest function such that $\deg(f) = \mathbf{0}^{[n]}$. By Theorem 4.7, we have an honest function g_0 such that $f <_E g_0$ and $f' \equiv_E g'_0$. Let $\mathbf{a}_\ell = \deg(g_0)$. Then we have $\mathbf{0}^{[n]} < \mathbf{a}_\ell < \mathbf{0}^{[n+1]} = \mathbf{a}'_\ell$. By Theorem 4.8, we have an honest function g_1 such that $g_1 <_E f'$ and $f'' \equiv_E g'_1$. Let $\mathbf{a}_h = \deg(g_1)$. Then we have $\mathbf{0}^{[n]} < \mathbf{a}_h < \mathbf{0}^{[n+1]}$ and $\mathbf{0}^{[n+2]} = \mathbf{a}'_h$. (The monotonicity of the jump operator assures that $\mathbf{a}_\ell < \mathbf{0}^{[n+1]}$ and that $\mathbf{0}^{[n]} < \mathbf{a}_h$.) $\Box$

Theorem 4.10 (Jump Inversion) *Let f and g_0 be honest functions such that $f' \le_E g_0 \le_E f''$. Then, there exists an honest function h such that $h \le_E f'$ and $h' \equiv_E g_0$.*

Proof. Since f is honest, we have $f'(x+1) \ge 2^{f'(x)}$ (and we also have $f'(x) \ge 2^x_{x+1}$). Furthermore, we can w.l.o.g. assume that we also have $g_0(x+1) \ge 2^{g_0(x)}$. If this should not the the case: Then, let $g_1(0) = g_0(0)$ and $g_1(x+1) = \max(2^{g_1(x)}, \max[g_0, f'](x+1))$. Then we obviously have $\max[g_0, f'] \le g_1$. Furthermore, for some $u, v \le x$ we have

$$g_1(x) \;=\; 2_u^{\max[g_0,f'](v)} \;\le\; 2_{\max[g_0,f'](x)+1}^{\max[g_0,f'](x)} \;\le\; \max[g_0,f']\max[g_0,f'](x)\;.$$

Thus, we have $\max[g_0, f'] \equiv_E g_1$ by the Growth Theorem, moreover, since $f' \le_E g_0$, we have $g_0 \equiv_E g_1$. This shows that we may replace g_0 by g_1 to ensure that $g_0(x+1) \ge 2^{g_0(x)}$.

We define the function g by recursion on its argument x. Let $g(0) = g_0(0)$ and let

$$g(x+1) \;=\; \begin{cases} f''(y+1) & \text{where } y \text{ is the least number s.t.} \\ & g(x) \le f''(y) < f''(y+1) < g_0(x+1) \\ & \\ g_0(x+1) & \text{if such } y \text{ does not exist.} \end{cases}$$

(Claim I) The function g is honest and
(a) $g \equiv_E g_0$
(b) $g(x) \le f''(y) \Rightarrow g(x+1) \le f''(y+1)$ for any $x, y \in \mathbb{N}$.

It is easy to see that g is an honest function, and Clause (b) of the claim is a straightforward consequence of the definition of g. We will now argue that $g(x+1) \le g_0(x+1) \le g(2x+1)$, and thus, Clause (a) follows by the Growth Theorem. It is obvious that $g(x+1) \le g_0(x+1)$. In order to verify that $g_0(x+1) \le g(2x+1)$, we observe that there might, or might not, exist $\ell > 0$ and a sequence $y_0, \dots, y_\ell$ such that

$$g(x) \;\le\; f''(y_0) \;\le\; f''(y_1) \;\le\; \dots \;\le\; f''(y_\ell) \;\le\; g_0(x+1) \;\le\; f''(y_\ell+1)\;.$$

If such a sequence does not exist, we have $g(x+1) = g_0(x+1)$ and thus also $g_0(x+1) \le g(2x+1)$. If such a sequence exists, we have $g(x+i) = f''(y_i)$ for $y = 1, \dots, \ell$ and $g(x+\ell+1) \ge g_0(x+1)$. Moreover, since $g_0(z) \le f''(z)$ holds for any z, the sequence $y_0, \dots, y_\ell$ cannot be very long, indeed, $\ell \le x$. Hence $g_0(x+1) \le g(x+\ell+1) \le g(x+x+1)$. This completes the proof of (Claim I).

For any injection ϕ, we define the function $\mathcal{I}_\phi$ by $\mathcal{I}_\phi(x) = \max(\mathcal{S}_\phi(x), 2^x)$ where $\mathcal{S}_\phi(0) = 0$ and

$$\mathcal{S}_\phi(x) = \begin{cases} \phi(i+1) & \text{if } x = \phi(i) \text{ for some } i \\ \mathcal{S}_\phi(x-1) & \text{otherwise} \end{cases}$$

when $x > 0$. The straightforward proof that $\mathcal{I}_\phi$ is an honest function whenever ϕ is an honest function, is left to the reader. We will prove that $\mathcal{I}'_g \equiv_E g$ and $\mathcal{I}_g \leq_E \mathcal{I}_{f''}$ and $\mathcal{I}_{f''} \leq_E f'$. Our theorem follows from these facts as we have $g_0 \equiv_E g$ by Claim I (a).

> **(Claim II)** For any honest function h where $h(x+1) \geq 2^{h(x)}$, we have
> (a) $h(x+1) = \mathcal{I}_h(h(x))$
> (b) $h(i) \leq x < h(i+1) \;\Rightarrow\; \mathcal{I}_h(x) = \max(h(i+1), 2^x)$.

Clause (a) of this claim holds since

$$\mathcal{I}_h(h(x)) \;=\; \max(\mathcal{S}_h(h(x)), 2^{h(x)}) \;=\; \max(h(x+1), 2^{h(x)}) \;=\; h(x+1)$$

and Clause (b) follows easily from Clause (a) and the definition of $\mathcal{I}_h$.

We will now prove that $\mathcal{I}'_g \equiv_E g$. Since $g(x+1) \geq 2^{g(x)}$, we have $g(x) = \mathcal{I}^x_g(g(0))$ by Claim II(a). Hence, it is easy to see that there exists fixed m, n such that $(\mathcal{I}'_g)^m(x) \geq g(x)$ and $g^n(x) \geq \mathcal{I}'_g(x)$ (recall that $\mathcal{I}'_g(x) = \mathcal{I}^{x+1}_g(x)$), and thus, the Growth Theorem yields $\mathcal{I}'_g \equiv_E g$.

Next we prove that $\mathcal{I}_g \leq_E \mathcal{I}_{f''}$. By the Growth Theorem, it suffices to prove $\mathcal{I}_g \leq \mathcal{I}^2_{f''}$. Pick and arbitrary x. If $\mathcal{I}_g(x) = 2^x$, we have $\mathcal{I}_g(x) \leq \mathcal{I}^2_{f''}(x)$ as f'' grows sufficiently fast. Now, assume $\mathcal{I}_g(x) \neq 2^x$. Fix the unique i and j such that $g(i) \leq x < g(i+1)$ and $f''(j) \leq g(i) < f''(j+1)$. Now

$$\begin{array}{rcll} \mathcal{I}^2_{f''}(x) & \geq & \mathcal{I}^2_{f''}(f''(j)) & \text{as } \mathcal{I}^2_{f''} \text{ is monotone and } x \geq f''(j) \\ & = & f''(j+2) & \text{Claim II (a)} \\ & \geq & g(i+1) & \text{Claim I (b) and } g(i) < f''(j+1) \\ & = & \mathcal{I}_g(x)\ . & \text{Claim II (b)} \end{array}$$

This proves $\mathcal{I}_g \leq_E \mathcal{I}_{f''}$.

Finally, we will prove that $\mathcal{I}_{f''} \leq_E f'$. Indeed, we will prove something stronger (given the Growth Theorem), namely that we have $\mathcal{I}_{h'} \leq h^2$ for any honest function h where $h(x+1) \geq 2^{h(x)}$. For such an h, we have

$$(*) \qquad \mathcal{I}_{h'}(h'(x)) \;=\; h'(x+1) \;=\; h^{x+2}(x+1) \;\leq\; h^2 h^{x+1}(x) \;=\; h^2(h'(x))\ .$$

Claim II assures that the first equality of (*) holds. The remaining relations of (*) hold trivially. Now, pick any x and fix the unique i such that $h'(i) \leq x < h'(i+1)$. If $\mathcal{I}_{h'}(x) = 2^x$, then $\mathcal{I}_{h'}(x) \leq h^2(x)$ holds trivially. If $\mathcal{I}_{h'}(x) \neq 2^x$, we have $\mathcal{I}_{h'}(x) = h'(i+1)$ by Claim II (b), and thus

$$\begin{array}{rcll} \mathcal{I}_{h'}(x) & = & h'(i+1) & \\ & = & \mathcal{I}_{h'}(h'(i)) & \text{Claim II (a)} \\ & \leq & h^2(h'(i)) & (*) \\ & \leq & h^2(x)\ . & \text{as } x \geq h'(i) \end{array}$$

This completes the proof of the theorem. □

Corollary 4.11 *Let* $\mathbf{a}$ *be a degree strictly between* $\mathbf{0}^{[n+1]}$ *and* $\mathbf{0}^{[n+2]}$. *Then, there exists a degree* $\mathbf{b}$ *strictly between* $\mathbf{0}^{[n]}$ *and* $\mathbf{0}^{[n+1]}$ *such that* $\mathbf{b}' = \mathbf{a}$.

Proof. Let f, g be honest function such that $\deg(f) = \mathbf{0}^{[n+1]}$ and $\deg(g) = \mathbf{a}$. We can w.l.o.g. assume that $f(x) \geq 2^x_{x+1}$. Now, Theorem 4.10 yields an honest function h such that $h \leq f$ and $h' \equiv g$. Let $\mathbf{b} = \deg(h)$. Then, we have $\mathbf{b}' = \mathbf{a}$, and by the monotonicity of the jump operator we also have $\mathbf{0}^{[n]} < \mathbf{b} < \mathbf{0}^{[n+1]}$. □

The next corollary follows straightforwardly from Corollary 4.9 and Corollary 4.11.

Corollary 4.12 (Low and High Degrees) *For any $n \in \mathbb{N}$, there exists a degree which is low$_n$, and there exists a degree which is high$_n$.*

Clause (i) of the next theorem is also proved in [**5**], whereas (ii) is stated as an open problem in [**5**].

Theorem 4.13 *(i) For any degrees $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a}' \cup \mathbf{b}' \leq (\mathbf{a} \cup \mathbf{b})'$. Moreover, there exist $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' \cup \mathbf{b}' = (\mathbf{a} \cup \mathbf{b})'$, and there exist $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' \cup \mathbf{b}' < (\mathbf{a} \cup \mathbf{b})'$. (ii) For any degrees $\mathbf{a}$ and $\mathbf{b}$, we have $\mathbf{a}' \cap \mathbf{b}' = (\mathbf{a} \cap \mathbf{b})'$.*

Proof. We start by proving (ii). Now, $\mathbf{a} \geq \mathbf{a} \cap \mathbf{b}$ holds in any lattice, and thus, by the monotonicity of the jump operator, we also have $\mathbf{a}' \geq (\mathbf{a} \cap \mathbf{b})'$. By the same token, we have $\mathbf{b}' \geq (\mathbf{a} \cap \mathbf{b})'$. Hence, as $\mathbf{a}' \cap \mathbf{b}'$ is the greatest lower bound of $\mathbf{a}'$ and $\mathbf{b}'$, we have $\mathbf{a}' \cap \mathbf{b}' \geq (\mathbf{a} \cap \mathbf{b})'$. We will now prove that $\mathbf{a}' \cap \mathbf{b}' \leq (\mathbf{a} \cap \mathbf{b})'$ also holds. Let f, g be honest functions such that $\mathbf{a} = \deg(f)$ and $\mathbf{b} = \deg(g)$. We have

$$
\begin{array}{rcll}
\min[f', g'](x) & = & \min(f'(x), g'(x)) & \\
& = & \min(f^{x+1}(x), g^{x+1}(x)) & \text{def. of the jump} \\
& \leq & \min[f, g]^{2(x+1)}(x) & \text{Lemma 3.3} \\
& \leq & \min[f, g]^{\min[f,g]'(x)+1+x+1}(x) & \\
& = & \min[f, g]^{\min[f,g]'(x)+1} \min[f, g]^{x+1}(x) & \\
& = & \min[f, g]^{\min[f,g]'(x)+1}(\min[f, g]'(x)) & \\
& = & \min[f, g]' \min[f, g]'(x) &
\end{array}
$$

and thus, by the Growth Theorem, we have $\min[f', g'] \leq_E \min[f, g]'$. This proves $\mathbf{a}' \cap \mathbf{b}' \leq (\mathbf{a} \cap \mathbf{b})'$, and (ii) follows.

We turn to the proof of (i). The proof of $\mathbf{a}' \cup \mathbf{b}' \leq (\mathbf{a} \cup \mathbf{b})'$ (for any degrees $\mathbf{a}, \mathbf{b}$) is symmetric to the proof of $\mathbf{a}' \cap \mathbf{b}' \geq (\mathbf{a} \cap \mathbf{b})'$ given above. Furthermore, it is obvious that there exists degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a}' \cup \mathbf{b}' = (\mathbf{a} \cup \mathbf{b})'$. The existence of $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' \cup \mathbf{b}' < (\mathbf{a} \cup \mathbf{b})'$ is a consequence of the following claim.

(Claim) For any degree $\mathbf{c} \geq \mathbf{0}'$, there exist degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{c} = \mathbf{a} \cup \mathbf{b} = \mathbf{a}' = \mathbf{b}'$.

By this claim, we have degrees $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that

$$\mathbf{a}' \cup \mathbf{b}' \;=\; \mathbf{c} \cup \mathbf{c} \;=\; \mathbf{c} \;<\; \mathbf{c}' \;=\; (\mathbf{a} \cup \mathbf{b})' \,.$$

To prove the claim, let $\mathbf{c}$ be a degree above $\mathbf{0}'$, and let f be an honest function such that $\deg(f') = \mathbf{c}$. Such a f exists by Theorem 4.10. Define the sequence $\{d_i\}_{i \in \mathbb{N}}$ by $d_0 = 0$ and $d_{i+1} = f'(d_i)$; define the functions G and H by $G(0) = H(0) = 0$ and, for $x > 0$, by

$$G(x) = \begin{cases} f'(x) & \text{if } x = d_{2i} \text{ for some } i \\ G(x-1) & \text{otherwise} \end{cases}$$

$$H(x) = \begin{cases} f'(x) & \text{if } x = d_{2i+1} \text{ for some } i \\ H(x-1) & \text{otherwise} \end{cases}$$

and, finally, let $g(x) = \max(f(x), G(x))$ and $h(x) = \max(f(x), H(x))$. It turns out that the claim holds when $\mathbf{a} = \deg(g)$ and $\mathbf{b} = \deg(h)$. The proof that this indeed is the case is nontrivial, and the details can be found in [**5**]. $\square$

An *intermediate* degree is a degrees below $\mathbf{0}'$ that, for any $n \in \mathbb{N}$, are neither low$_n$ nor high$_n$. We conclude this section by a theorem stating the existence of an intermediate degree.

Theorem 4.14 *There exists a degree* $\mathbf{a}$ *such that, for any* $n \in \mathbb{N}$*, we have* $\mathbf{0}^{[n]} < \mathbf{a}^{[n]} < \mathbf{0}^{[n+1]}$.

Proof. Let $f(x) = 2^x$. Define the sequence $\{d_i\}_{i\in\mathbb{N}}$ by $d_0 = 0$ and $d_{i+1} = f^{[d_i]}(d_i)$; define the function G by $G(0) = 0$ and, for $x > 0$, by

$$G(x) = \begin{cases} f'(x) & \text{if } d_{3i} \leq x < d_{3i+1} \text{ for some } i \\ G(x-1) & \text{otherwise} \end{cases}$$

and let $g(x) = \max(f(x), G(x))$. Now, g is an honest function, and $f \leq g \leq f'$. By the Growth Theorem, we have

$$\mathbf{0} \;=\; \deg(f) \;\leq\; \deg(g) \;\leq\; \deg(f') \;=\; \mathbf{0}' \,.$$

By the monotonicity of the jump operator, we have $\mathbf{0}^{[n]} \leq \deg(g)^{[n]} \leq \mathbf{0}^{[n+1]}$ for any $n \in \mathbb{N}$. It remains to prove that $\deg(g)^{[n]} \not\leq \mathbf{0}^{[n]}$ and $\mathbf{0}^{[n+1]} \not\leq \deg(g)^{[n]}$. The details can be found in [**7**]. $\square$

5 On cupability and capability

Definition 5.1 *A degree* $\mathbf{a}$ cups (up) to *a degree* $\mathbf{b}$ *if there exists* $\mathbf{c}$ *such that* $\mathbf{c} < \mathbf{b}$ *and* $\mathbf{a} \cup \mathbf{c} = \mathbf{b}$. *A degree* $\mathbf{a}$ caps (down) to *a degree* $\mathbf{b}$ *if there exists* $\mathbf{c}$ *such that* $\mathbf{b} < \mathbf{c}$ *and* $\mathbf{a} \cap \mathbf{c} = \mathbf{b}$.

Next we define the binary relation $\ll$ *on honest functions. A function* $\rho : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *is a* universal *for an honest degree* $\mathbf{a} = deg(f)$ *if for every* $\xi : \mathbb{N} \to \mathbb{N}$ *such that* $\xi \leq_E f$*, we have* $\xi(x) = \rho(n, x)$ *for some* $n \in \mathbb{N}$*. The relation* $f \ll g$ *holds if there exists a universal function* ρ *for the degree* $deg(f)$ *such that* $\rho \leq_E g$*. We will also use* $\ll$ *to denote the corresponding relation on honest degrees.* $\square$

The situation $\mathbf{a} \ll \mathbf{b}$ implies that $\mathbf{a} < \mathbf{b}$, but there exist degrees $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} < \mathbf{b}$ and $\mathbf{a} \not\ll \mathbf{b}$. The next lemma gives a characterisation of the $\ll$-relation.

Lemma 5.2 *Let* g *and* f *be honest functions. Then (1) and (2) are equivalent.*

(1) $g \ll f$
(2) *there exists* m *such that, for any* k*, we have we have* $g^k(x) < f^m(x)$ *for all but finitely many* x.

Proof. To prove this lemma, we need a refined version of the Kleene Normal Form Theorem. We assume the reader is familiar with the *computable functions*, *indexes* for computable functions, *computation trees* and other well-known concepts in computability theory. When e is an index for the computable function f, we adopt the traditional abuse of notation and write $\{e\}(\vec{x})$ both for (i) the computation of $f(\vec{x})$ associated with e and for (ii) the eventual result of the computation. Let $\mathcal{U}$ be a function such that $\mathcal{U}(\langle x_1, \ldots, x_m\rangle) = x_m$, i.e. a function giving the last coordinate of a sequence number.

Let $\mathcal{T}$ be the Kleene predicate, i.e. the predicate $\mathcal{T}(e, \langle x_1, \ldots, x_n \rangle, t)$ holds iff t is a computation tree for $\{e\}(x_1, \ldots, x_n)$. The relation $\mathcal{T}$ is elementary, so is the function $\mathcal{U}$, and for each total computable function ϕ we have

$$\phi(x_1, \ldots, x_n) \;=\; \{e\}(x_1, \ldots, x_n) \;=\; \mathcal{U}(\mu z[\mathcal{T}(e, \langle x_1, \ldots, x_n \rangle, z)])$$

when e is a computable index for ϕ.

> **Claim** (Normal Form Theorem). An n-ary function ψ is elementary in an honest function f iff there exist a recursive index e for ψ and a fixed number k such that
>
> $$\{e\}(x_1, \ldots, x_n) \;=\; \mathcal{U}(\mu y \leq f^k(\max(x_1, \ldots, x_n))[\mathcal{T}(e, \langle x_1, \ldots, x_n \rangle, y)]) \;.$$

We sketch a proof of this claim: Assume

$$\psi(\vec{x}) \;=\; \{e\}(\vec{x}) \;=\; \mathcal{U}(\mu y \leq f^k(\max(\vec{x}))[\mathcal{T}(e, \langle \vec{x} \rangle, y)]) \;.$$

The predicate $\mathcal{T}$ is elementary, and $\mathcal{U}$ and max are elementary functions. The elementary functions are closed under composition and the bounded μ-operator. Thus, ψ is elementary in f. To prove the other direction of the equivalence, assume that ψ is elementary in the honest function f. Then, ψ can be build from the functions $0, \mathcal{S}, \mathcal{I}_i^n$, max and f by composition and bounded primitive recursion. Complete the proof of the claim by induction on such a build-up of ψ. (The details can be found in [**6**].)

We will now turn to the proof of the lemma. Fix m such that, for any k, we have $g^k(x) < f^m(x)$ for all but finitely many x. Then, for every k, there exists $n_k \in \mathbb{N}$ such that

$$(*) \qquad g^k(x) \;<\; n_k + f^m(x)$$

holds for all x. Let ξ be any unary function elementary in g. By the claim we have an index e for ξ, an elementary predicate $\mathcal{T}_1$, an elementary function $\mathcal{U}$ and a fixed $\ell \in \mathbb{N}$ such that

$$\xi(x) \;=\; \mathcal{U}((\mu t \leq g^\ell(x))[\mathcal{T}_1(e, x, t)]) \;.$$

By (*), we have $n_\ell \in \mathbb{N}$ such that

$$\xi(x) \;=\; \mathcal{U}((\mu t \leq g^\ell(x))[\mathcal{T}_1(e, x, t)]) \;=\; \mathcal{U}((\mu t \leq n_\ell + f^m(x))[\mathcal{T}_1(e, x, t)]) \;.$$

Let $\rho(\langle e, n \rangle, x) = \mathcal{U}((\mu t < n + f^m(x))[\mathcal{T}_1(e, x, t)])$. Then, we have $\rho \leq_E f$, and for every unary function ξ elementary in g, there exists n such that $\xi(x) = \rho(n, x)$. This proves that (1) implies (2).

Assume $g \ll f$. Then, there exists a function ρ such that ρ is a universal function for $\deg(g)$ and $\rho \leq_E f$. Let $\psi(x) = (\max_{i \leq x} \max_{j \leq x} \rho(i, j)) + 1$. Then, we have $\psi \leq_E f$, and hence, there exists m such that $\psi(x) \leq f^m(x)$. It is easy to see that for any unary function ϕ elementary in g, we have $\phi(x) < \psi(x) \leq f^m(x)$ for all but finitely many x. Thus, for any k, as $g^k \leq_E g$, we have $g^k(x) < f^m(x)$ for all but finitely many x. This proves that (2) implies (1). $\square$

The next theorem was proved for the first time in [**8**].

Theorem 5.3 *If* $\mathbf{0} \ll \mathbf{a} < \mathbf{b}$*, then* $\mathbf{a}$ *cups to* $\mathbf{b}$.

Proof. Let f and g be honest functions such that $\deg(f) = \mathbf{a}$, and $\deg(g) = \mathbf{b}$, and $f \leq g$. Define the sequence $\{d_i\}_{i \in \mathbb{N}}$ by $d_0 = 0$; $d_{2i+1} = g(d_{2i})$; and $d_{2i+2} = f(d_{2i+1})$.

Furthermore, define the function h by $h(x) = \max(H(x), 2^x)$ where $H(0) = 0$ and, for $x > 0$

$$H(x) = \begin{cases} g(x) & \text{if } x = d_{2i} \text{ for some } i \\ H(x-1) & \text{otherwise.} \end{cases}$$

It is possible to prove that h is an honest function such that $\max[f, h] \equiv_E g$ and $g \not\leq_E h$. The details can be found in [**8**]. $\square$

We have tried hard to strengthen Theorem 5.3 by proving that $\mathbf{a}$ cups up to $\mathbf{b}$ whenever $\mathbf{0} < \mathbf{a} < \mathbf{b}$. We have not succeeded, and thus it remains an open problem if there exist degrees other than $\mathbf{0}$ that do not cup up to degrees above them. However, with a possible exceptions of some degrees not being $\ll$-above $\mathbf{0}$, any degree cups up to any degree above it, and thus, "cups up to" is a not a very restrictive relation. We will see that the relation "caps down to" is a far more restrictive.

Lemma 5.4 *Let g, f be honest functions such that f caps to g and $g \leq f$. Then, there exist a fixed $c \in \mathbb{N}$ such that for each k, we have $f^k(x) \leq g^{ck}(x)$ for infinitely many x.*

Proof. Since f caps to g we have an honest h such that $\min[f, h] \leq_E g$. By the Growth Theorem, we can fix a $c \in \mathbb{N}$ such that $\min[f, h] \leq g^c$. Now, as $\min[f, h]$ and g are monotone, we also have $\min[f, h]^k \leq g^{ck}$ (for any k). Moreover, as we have assumed $g \leq f$, we have $\min[f, h]^k \leq g^{ck} \leq f^{ck}$ (for any k). As f caps to g by h, we have $h \not\leq_E f$, and thus, by the Growth Theorem, for any $c, k \in \mathbb{N}$ we have infinitely many values $x_0, x_1, x_2, \ldots$ such that $f^{ck}(x_i) < h(x_i)$. For each x_i of these values, we have

$$(*) \qquad \min[f, h]^k(x_i) \;\leq\; g^{ck}(x_i) \;\leq\; f^{ck}(x_i) \;<\; h(x_i)\,.$$

This entails that $\min[f, h]^k(x_i) = f^k(x_i)$. If not, (*) yields a contradiction. Thus, (*) entails that $f^k(x_i) \leq g^{ck}(x_i)$ for each x_i in the sequence $x_0, x_1, x_2, \ldots$. $\square$

Theorem 5.5 *If $\mathbf{a} \ll \mathbf{b}$, then $\mathbf{b}$ does not cap to $\mathbf{a}$.*

Proof. Assume that $\deg(g) = \mathbf{a} \ll \mathbf{b} = \deg(f)$ and that $\mathbf{b}$ caps to $\mathbf{a}$. We can w.l.o.g. assume $g \leq f$. Since $\mathbf{a} \ll \mathbf{b}$, Lemma 5.2 yields a fix m such that for any k, we have $g^k(x) < f^m(x)$ for all but finitely many x. Since $\mathbf{b}$ caps to $\mathbf{a}$, Lemma 5.4 yields a fixed c such that for each k, we have $f^k(x) \leq g^{ck}(x)$ for infinitely many x. This is a contradiction. $\square$

It is natural to ask whether the converse of Theorem 5.5 also holds, that is, do we have $\mathbf{a} \ll \mathbf{b}$ if, and only if, $\mathbf{b}$ does not cap to $\mathbf{a}$? (This was stated as an open problem in [**7**].) The next theorem gives a negative answer to this question.

Theorem 5.6 *There exist degrees $\mathbf{a} < \mathbf{b}$ such that $\mathbf{b}$ does not cap to $\mathbf{a}$ even if we have $\mathbf{a} \not\ll \mathbf{b}$.*

Proof. Let f be an honest function such that $f(x) \geq 2^x_x$. We will construct an honest function g and prove the two following claims.

(Claim I) For any m, we have $g^{m^2}(x) = f^m(x)$ for infinitely many x.

(Claim II) For any m, we have $g^{m^2}(x) < f^{3m+1}(x)$ for all but finitely many x.

Let $\nu(k)$ equal 1 plus the exponent of 2 in the prime factorisation of $k + 2$. Thus, ν is an elementary function. (Any elementary function ϕ such that the set $\{x \mid \phi(x) = n\}$ is infinite for all $n > 0$, could replace ν in this proof.) For each $k \in \mathbb{N}$, we will define a sequence $d_{k,0} < d_{k,1} < \ldots < d_{k,\nu(k)^2}$. Moreover, for each k, we will have $d_{k,\nu(k)^2} < d_{k+1,0}$. Let $d_{0,0} = 0$. For each $j \in \{1, \ldots, \nu(k)^2\}$, let

$$d_{k,j} = \begin{cases} f(d_{k,j-\nu(k)}) & \text{if } \nu(k) \text{ divides } j \\ 2^{d_{k,j-1}} & \text{otherwise} \end{cases}$$

and let $d_{k+1,0} = f'(d_{k,\nu(k)^2})$. Furthermore, let

$$G(x) = \begin{cases} d_{k,i+1} & \text{if } d_{k,i} \leq x < d_{k,i+1} \text{ for some } k, i \\ d_{k,\nu(k)^2} & \text{if } d_{k,\nu(k)^2} \leq x < d_{k+1,0} \text{ for some } k, i \end{cases}$$

and let $g(x) = \max(2^x, G(x))$. This completes the construction of g. The reader should note the following properties of g (and f):

(P1) $g(d_{k,i}) = d_{k,i+1}$ for any k and any $i < \nu(k)^2$
(P2) for any k and any $i < \nu(k)^2$, we have $g(d_{k,i}) = f(d_{k,i})$ if $\nu(k)$ divides i
(P3) for any k and any $i < \nu(k)^2$, we have $g(d_{k,i}) = 2^{d_{k,i}}$ if $\nu(k)$ does not divide i
(P4) $g^{\nu(k)^2}(d_{k,0}) = d_{k,\nu(k)^2} = f^{\nu(k)}(d_{k,0})$ for any k
(P5) for any m, we have $g^m(d_{k,\nu(k)^2}) = 2_m^{d_{k,\nu(k)^2}} < d_{k+1,0}$ for all but finitely many k.

These five properties is more or less straightforward consequences of the construction of g, in particular, to see that (P5) holds, note that $d_{k+1,0} = f'(d_{k,\nu(k)^2})$ and $f(x) \geq 2^x_x$.

(Claim I) follows straightaway from (P4). For any m we have $g^{m^2}(d_{k,0}) = f^m(d_{k,0})$ for each of the infinitely many k's such that $\nu(k) = m$. We turn to the proof of (Claim II). The proof splits into the two cases: the case when x lies in an interval of the form $d_{k,0}, \ldots, d_{k,\nu(k)} - 1$, and the case when x lies in an interval of the form $d_{k,\nu(k)}, \ldots, d_{k+1,0} - 1$.

We will first prove that we have $g^{m^2}(x) < f^{3m+1}(x)$ when x is sufficiently large and lies in an interval of the form $d_{k,0}, \ldots, d_{k,\nu(k)} - 1$. The proofs splits into the the two sub-cases $m \geq \nu(k)$ and $m < \nu(k)$. First, assume that $m \geq \nu(k)$. We have

$$\begin{array}{rcll}
f^{3m+1}(x) & = & f^{(3m+1)-\nu(k)} f^{\nu(k)}(x) & \\
 & \geq & f^{(3m+1)-\nu(k)} f^{\nu(k)}(d_{k,0}) & f \text{ is monotone} \\
 & = & f^{(3m+1)-\nu(k)}(d_{k,\nu(k)^2}) & \text{(P4)} \\
 & > & f(d_{k,\nu(k)^2}) & \text{as } m \geq \nu(k) \\
 & \geq & 2^{d_{k,\nu(k)^2}}_{d_{k,\nu(k)^2}} & \text{as } f(x) \geq 2^x_x \\
 & \geq & 2^{d_{k,\nu(k)^2}}_{m^2} & x \text{ is large} \\
 & = & g^{m^2}(d_{k,\nu(k)^2}) & \text{(P5) and } x \text{ is large} \\
 & \geq & g^{m^2}(x)\ . & g \text{ is monotone}
\end{array}$$

Next, assume that $m < \nu(k)$. Fix the unique i such that $d_{k,i} \leq x < d_{k,i+1}$. Since $m < \nu(k)$, there will be at most one number j in the interval $i, \ldots, \min(i + m, \nu(k)^2)$ such that $\nu(k)$ divides j. Hence, by (P2), (P3) and (P5), there exist m_0, m_1 such that

$$(\dagger) \qquad g^m(x) \ \leq \ 2^{f(2^x_{m_1})}_{m_0} \ \leq \ f^3(x)\ .$$

Furthermore, g is monotone and $x \leq d_{k,\nu(k)^2}$, and then, by (P5), we have

$$(\ddagger) \qquad g^{m^2}(x) \;\leq\; g^{m^2}(d_{k,\nu(k)^2}) \;< d_{k+1,0}$$

for all but finitely many x. It follows from (†) and (‡), we have $g^{m^2}(x) < f^{3m+1}(x)$ for all sufficiently large x.

The reader is invited to verify that we also have $g^{m^2}(x) < f^{3m+1}(x)$ for sufficiently large x lying in intervals of the form $d_{k,\nu(k)}, \ldots, d_{k+1,0} - 1$. To verify this, note that for any x in such an interval we have $g(x) = 2^x$ whereas $f(x) \geq 2^x_x$. This completes the proof of (Claim II).

We will briefly now argue that g is honest an honest function. The function f is honest by assumption. First we argue that $d_{k,j} = x$ is an elementary relation in k, j, x. Let $a \mid b$ denote the relation "a divides b". This relation is elementary. We have

$$\begin{aligned} d_{k,j} = x \;\; \Leftrightarrow \qquad & \\ & \big(j \neq 0 \;\; \wedge \;\; \nu(k) \mid j \;\; \wedge \;\; \exists x_0 < x \,[\, d_{k,j-\nu(k)} = x_0 \;\; \wedge \;\; f(x_0) = x \,]\big) \;\; \vee \\ & \quad (j \neq 0 \;\; \wedge \;\; \neg\, \nu(k) \mid j \;\; \wedge \;\; \exists x_0 < x \,[\, d_{k,j-1} = x_0 \;\; \wedge \;\; 2^{x_0} = x \,]) \;\; \vee \\ & \qquad \big(j = 0 \;\; \wedge \;\; \exists x_0 < x \,[\, d_{k,\nu(k)^2} = x_0 \;\; \wedge \;\; 2^{x_0} = x \,]\big) \;\; \vee \\ & \qquad\qquad\qquad\qquad\qquad\qquad (\, k = 0 \;\; \wedge \;\; j = 0 \;\; \wedge \;\; x = 0 \,) \; . \end{aligned}$$

This can be viewed as a recursive definition of $d_{k,j} = x$. All the functions, relations and operations involved are elementary. Thus, we have defined the relation $d_{k,j} = x$ by a recursion scheme of the form

$$R(k, j, x) \Leftrightarrow \phi(R(k_0, j_0, x_0), R(k_1, j_1, x_1), R(k_2, j_2, x_2))$$

where ϕ is an elementary predicate and $k_0, k_1, k_2 \leq k$; $j_0, j_1, j_2 \leq k$; and $x_0, x_1, x_2 \leq x$. The elementary predicates are closed under such a recursion scheme, and hence, $d_{k,j} = x$ is an elementary relation. Thus, $\exists k, j \leq x[d_{k,j} = x]$ is an elementary predicate. Once we have realised that this predicate is elementary, it becomes easy to see that g has elementary graph. Obviously, g is monotone and dominates 2^x. Thereby, g is honest.

We will now prove the theorem. We have $g \leq_E f$ by the Growth Theorem since $g \leq f$. Let m be any number. Pick x such that $x > m$ and $x = d_{k,\nu(k)^2}$ for some k. By (P5), we have $g^m(x) = 2^x_m < 2^x_x \leq f(x)$. Hence, we have $f \not\leq_E g$ by the Growth Theorem. This proves $g <_E f$. (Claim I) says that for any m there exist infinitely many x such that $g^{m^2}(x) = f^m(x)$. This entails that there cannot exist a fixed number n such that we for any m have $g^m(x) < f^n(x)$ for all but finitely many x. Thus, we have $g \not\ll f$ by Lemma 5.2. Finally, (Claim II) and Lemma 5.4 entail that f does not cup to g, and then, our theorem holds when $\mathbf{a} = \deg(g)$ and $\mathbf{b} = \deg(f)$. $\square$

6 Controllable irreducibility and the pendulum theorem

Definition 6.1 *A sequence of natural numbers $\{d_i\}_{i \in \mathbb{N}}$ is elementary if the the relation $d_i = y$ is elementary. An honest function f is* controllably irreducible *to a an honest function g if there exists an elementary sequence $d_0 < d_1 < d_2 < \ldots$ such that for any k we have $g^k(d_i) < f(d_i)$ for all but finitely many i.* $\square$

In the next theorem we assume that a function f is controllably irreducible to a function h. We do not know how to prove this theorem if we only assume that f is irreducible to h.

Theorem 6.2 (Pendulum) *Let f, g and h be honest functions such that f is controllably irreducible to h and $g <_E f \leq_E g'$. Then there exists an honest function g_0 such that (i) $g <_E g_0 <_E f$ (and f is controllably irreducible to g_0), (ii) $g_0 \not\leq_E h$ and (iii) $g_0' \equiv_E g'$.*

Proof. Let $e_0 < e_1 < e_2 < \ldots$ be an elementary sequence such that for any k we have $h^k(e_i) < f(e_i)$ for all sufficiently large e_i. Such a sequence exists since f is controllably irreducible to h. We construct the sequence $d_0 < d_1 < d_2 < \ldots$ by letting $d_0 = 0$ and $d_{i+1} = e_j$ where where e_j is the least element in the sequence $e_0 < e_1 < e_2 < \ldots$ such that

$$g'g'g'(d_i) < e_j \ \wedge \ \exists y \leq e_j \exists x \leq y \, [\, f(x) = y \ \wedge \ g^i(x) < y \,] \ .$$

The sequence $\{d_i\}_{i \in \mathbb{N}}$ is well defined as $f \not\leq_E g$, and by the Growth Theorem, for each i there exists infinitely many x such that $g^i(x) < f(x)$. Moreover, the sequence is elementary as d_{i+1} is defined from d_i by elementary operations.

Let $g_0(x) = \max(\mathcal{S}_f(x), g(x))$ where $\mathcal{S}_f(0) = 0$ and

$$\mathcal{S}_f(x) = \begin{cases} f(x) & \text{if } x = d_i \text{ for some } i \\ \mathcal{S}_f(x-1) & \text{otherwise} \end{cases}$$

when $x > 0$. Since that f and g are honest and $\{d_i\}_{i \in \mathbb{N}}$ is elementary, it is straightforward to verify that that g_0 is an honest function.

We will first prove that Clause (i) of the Theorem holds. Since $g <_E f$, we can w.l.o.g. assume that $g(x) \leq f(x)$. This entails that we also have $g_0(x) \leq f(x)$, and thus, $g_0 \leq_E f$ follows by the Growth Theorem. Moreover, we have constructed g_0 such that we for each k have infinitely many x such that $g_0^k(x) < f(x)$, and thus, again by the Growth Theorem, we have $f \not\leq_E g_0$. This proves that $g_0 <_E f$. Obviously, we also have $g <_E g_0$. Thus, (i) holds.

It is easy to prove that (ii) holds. In order to see that $g_0 \not\leq_E h$, just observe that for any k we have $g_0(d_i) = f(d_i) > h^k(d_i)$ for all but finitely many d_i, and then, use the Growth Theorem. This completes the proofs of (ii).

> **(Claim)** Let $g'(d_i) \leq x \leq g'g'(d_i)$. Then, $g_0^y(x) = g^y(x)$ whenever $y \leq x$.

It should not be hard to see that this claim holds: Observe that

(a) $g_0(z) = g(z)$ for any z in the interval $g'(d_i), \ldots, d_{i+1} - 1$
(b) $g^y(x) < g'(x) < g'(g'g'(d_i)) \leq d_{i+1}$.

The claim follows easily from (a) and (b).

Next we prove that $g_0'(x) \leq g'g'g'(x)$. Pick an arbitrary x and fix i such that $d_i \leq x < d_{i+1}$. There exists a maximal number z such that $z \leq x + 1$ and

$$g_0'(x) \ = \ g_0^{x+1}(x) \ = \ g_0^{(x+1)-z} g^z(x) \ .$$

If $z = x + 1$, then $g_0'(x) \leq g'g'g'(x)$ holds trivially. Assume $z < x + 1$. Now, $z < x + 1$ implies that $d_{i+1} \leq g^z(x)$. This is easily verified by inspecting the definition of g_0. Furthermore, note that we can assume that $f(x) \leq g'(x)$. There will be no loss of generality to assume this as $f \leq_E g'$. We have

$$\begin{array}{rcll} g_0'(x) & = & g_0^{(x+1)-z} g^z(x) & \\ & = & g_0^{x-z} g_0 g^z(x) & \\ & = & g_0^{x-z} \max(\mathcal{S}_f(g^z(x)), gg^z(x)) & \text{def. of } g_0 \\ & \leq & g_0^{x-z} \max(f(g^z(x)), gg^z(x)) & \text{def. of } \mathcal{S}_f \\ & \leq & g_0^{x-z} \max(f(g'(x)), g'(x)) & \text{def. of } g' \text{ and } z \leq x \\ & \leq & g_0^{x-z} g'g'(x) \ . & \text{since } f(x) \leq g'(x) \end{array}$$

This proves that $g_0'(x) \leq g_0^{x-z} g'g'(x)$ for some $z \leq x$ such that $d_{i+1} \leq g^z(x)$. We also have $g'(d_{i+1}) \leq g'g^z(x) \leq g'g'(x) \leq g'g'(d_{i+1})$, and hence, $g_0'(x) \leq g'g'g'(x)$ follows by (Claim).

This proves that $g_0'(x) \leq g'g'g'(x)$ holds for any x. By the the Growth Theorem, we have $g_0' \leq_E g'$. Furthermore, it is easy to see that $g \leq_E g_0$, and hence, we have $g' \leq_E g_0'$ by the monotonicity of the jump operator. Thus, $g_0 \equiv_E g$. This completes the proof of (iii). □

Before we investigate the notion of controllable irreducibility further, we will discuss what it should mean for a degree to be controllably irreducible to another degree: The Growth Theorem entails that if f is controllably irreducible to g, then f is controllably irreducible to any h elementary in g. So we can say that f is controllably irreducible to $\deg(g)$ if f is controllably irreducible to some, or equivalently all, representative(s) in $\deg(g)$. The same cannot be said when replacing f by its degree. This motivates the next definition.

Definition 6.3 *A degree* $\mathbf{a}$ *is* controllably irreducible *to a degree* $\mathbf{b}$ *when some function in* $\mathbf{a}$ *is controllably irreducible to some, or equivalently all, function(s) in* $\mathbf{b}$*. A degree* $\mathbf{a}$ *is* not controllably irreducible *to a degree* $\mathbf{b}$ *when no function in* $\mathbf{a}$ *is controllably irreducible to some, or equivalently all, function(s) in* $\mathbf{b}$*. A degree* $\mathbf{b}$ *is* slightly above *a degree* $\mathbf{a}$ *when* $\mathbf{a} < \mathbf{b}$ *and* $\mathbf{b}$ *is not controllably irreducible to* $\mathbf{a}$*.* □

The next theorem entails that if there exists one degree that is slightly above a degree $\mathbf{a}$, then there will be a lot of degrees slightly above $\mathbf{a}$.

Theorem 6.4 *Let* $\mathbf{b}$ *be slightly above* $\mathbf{a}$*, and let* $\mathbf{a} \leq \mathbf{c}_i \leq \mathbf{b}$ *for* $i = 1, 2$*. Then,* $\mathbf{c}_2$ *cannot be controllably irreducible to* $\mathbf{c}_1$*.*

Proof. Assume that $\mathbf{c}_2$ is controllably irreducible to $\mathbf{c}_1 = \deg(g)$. Then, there exist $f \in \mathbf{c}_2$ and and elementary sequence $d_0 < d_1 < d_2 < \ldots$ such that for any k we have $g^k(d_i) < f(d_i)$ for all but finitely many i. Let $\mathbf{a} = \deg(h_1)$ and $\mathbf{b} = \deg(h_2)$. We can w.l.o.g. assume that $h_1 \leq g$ and $f \leq h_2$, and then, for any k, we have $h_1^k(d_i) < h_2(d_i)$ for all but finitely many i. This contradicts that $\mathbf{b}$ is slightly above $\mathbf{a}$. □

The next theorem requires proof techniques based on enumerations and diagonalisations. This is the first result we prove on the structure of honest elementary degrees that require such techniques.

Theorem 6.5 *There exists a degree that is slightly above* $\mathbf{0}$*.*

Proof. We will construct an honest function f such that $\deg(f)$ is not controllably irreducible to $\mathbf{0} = \deg(2^x)$. We have to prove that no function in $\deg(f)$ is controllably irreducible to 2^x. By the Growth Theorem, it is sufficient to prove that no finite iterate of f is controllably irreducible to 2^x. Besides, we have to prove that f is not elementary, that is, we have to prove that no fixed iterate of 2^x dominates f.

Thus, on the one hand, f will have to grow somewhat fast: at some point it must be greater than any given iterate of 2^x. On the other hand, we must make certain that no elementary sequence $d_0 < d_1 < d_2 < \ldots$ is a witness to the undesired controlled irreducibility. That involves diagonalising against all such possible sequences. Furthermore, this diagonalisation must work for all finite iterations of f.

To improve the readability, we will throughout this proof use the notation $2_x(y)$ in place of 2^y_x.

We need a master list of sequences $d_0 < d_1 < d_2 < \ldots$. There is no good elementary listing of all such total sequences, but there is one if we allow for partial (finite) sequences, as follows. Let $t_0, t_1, t_2 \ldots$ be a listing of all elementary functions in two variables induced by using some primitive recursive coding of the base functions and operations allowed in the definition of elementarity. There is no universal elementary function for this listing; that is, the relation $t_i(x, y) = z$ is not elementary. However, because of the simplicity of the coding, one can code a particular computation as an integer and use that the relation

$$q \text{ bounds a witness that } t_i(x, y) = z$$

is elementary. For every elementary sequence $d_0 < d_1 < d_2 < \ldots$ there is an i such that $t_i(x, y)$ is the characteristic function of the relation $d_x = y$. In the other direction, given i and q, it is elementary to see whether t_i looks like the characteristic function of such a sequence when considering only witnesses beneath q. If t_i is not the characteristic function of such a sequence, then eventually there will be a witness beneath q showing that. Let T_i be the sequence so induced by t_i, either an infinite sequence $d_0 < d_1 < d_2 < \ldots$ if t_i is a good characteristic function, or a finite sequence if not. We will have to diagonalise against T_i if it is total without knowing whether it is total.

We now define a function f as follows. At stage n we will define f on the n^{th} interval $I_n = [x_n, x_{n+1})$. To start, put $I_0 = \{0\}$, and $f(0) = 2$. We use an auxiliary function $L(n) \subseteq n$, which tells us at stage $n+1$ which T_i's (for $i < n$) do *not* need to be attended to. (One problem is that some T_i might always demand attention. Once it gets attended to, it gets put on the list L, allowing other requirements to be met. It will eventually be taken off the list and, if it remains active, will then be attended to again.) To start, put $L(0) = \emptyset$. Suppose inductively that we have defined the set $L(n-1)$ and the function f up to x_n. We will define I_n (i.e. determine x_{n+1}), and f on I_n, and $L(n)$, in several steps. First consider $J_{n,0} = [x_n, 2_n(x_n)]$ (the first sub-interval of I_n). We would like to pick a T_i to work on, if possible. So consider all $j < n$ not in $L(n-1)$ for which some $z \in J_{n,0}$ is in the range of T_j. Choose the pair j, z for which $y = 2_j(z)$ is less than $2_{n+1}(x_n)$, bounds a witness that z is in the range of T_j, and is the minimal such number; if there are several choices giving the same value, pick the one with j minimal. We call this value of j the active index for the interval I_n. Then we put $f(x) = \max(y, 2^x)$ on $[x_n, 2_n(x_n)]$. The outcome of this action is that f grows reasonably fast (at least as fast as 2_j) from x_n to that z, and no faster than that afterwards for a while. We set $L(n) = (L(n-1) \cup \{j\}) \setminus \{0, \ldots, j-1\}$: since j just got attended to, it can be ignored for a while, yet allows smaller requirements to receive attention. If no such pair j, z exists, we put $f(x) = 2_{n+1}(x_n)$ on $J_{n,0}$.

Now we need to consider iterations of f, and make sure that they grow slowly. We will define $J_{n,k}$ and X_k inductively on k. $J_{n,0}$ is already defined; let $X_0 = \{0, \ldots, n-1\}$.

Suppose we have already defined the interval $J_{n,k-1} = [x_{n,k-1}, x_{n,k})$ and f on $J_{n,k-1}$. Then we put $J_{n,k} = [x_{n,k}, 2_n(x_{n,k}))$ and set $f(x) = 2^x$ on this interval. If there exists some $i \in X_{k-1}$ such that T_i has a value in $J_{n,k-1}$, then we put $X_k = X_{k-1} \setminus \{i\}$. For some k there will be so such i (as X_0 is finite and the X-sequence is monotonically shrinking). When that happens, put $x_{n+1} = x_{n,k+1}$. That completes stage n.

This completes the definition of f. To complete the proof of the theorem, we will prove that

(1) f is honest
(2) f is not elementary
(3) no function in $\deg(f)$ is controllably reducible to a function in $\mathbf{0}$.

First we prove (1). We obviously have $f(x) \geq 2^x$ for every x. Furthermore, each interval I_n contains one subinterval $[x_n, q]$ (namely for $q = 2_{j-1}(z)$), on which f is constant and equal to 2^q, and one subinterval $[z+1, x_{n+1}]$, on which f equals 2^x. Hence in the interior of each I_n f is non-decreasing. Finally $f(x_{n+1}-1) = 2^{x_{n+1}-1} < 2^{x_{n+1}} \leq f(x_{n+1})$, hence f is globally non-decreasing. It remains to show that the graph of f is elementary. The auxiliary function L can be encoded into integers up to 2^n, so for a given x we can decide what kind of interval x is in, and which values $j \in \{1, \ldots, n\} \setminus L(n-1)$ are possible. In particular for each x we can compute the value x_n for which $x_n \leq x < x_{n+1}$, and it suffices to compute $f(x_n)$ from these data. This is possible because we have $f(x_n) = y$ iff

$$\begin{aligned}&(\exists j \leq n)(\exists \xi, \zeta < y)[\, j \notin L(n-1) \ \wedge\ t_j(\xi, \zeta) = 1 \ \wedge\ 2_j(\zeta) = y \,] \ \wedge \\ &\qquad (\forall y' < y) \neg (\exists j \leq n)(\exists \xi, \zeta < y)[\, j \notin L(n-1) \ \wedge\ t_j(\xi, \zeta) = 1 \ \wedge\ 2_j(\zeta) = y \,].\end{aligned}$$

Hence, the graph of f is elementary. This proves that f is an honest function.

We turn to the proof of (2). We have to show that for every k there exists some x, such that $f(x) > 2_k(x)$. For this it is sufficient to show that for every k there exists some $\ell > k$ such that ℓ is active in some interval I_n. There are infinitely many simple ways to describe the function $x \mapsto 2^x$, so choose some term t_ℓ describing this function with $\ell > k$ such that $2_\ell(x)$ bounds a witness that $T_\ell(x) = 2^x$. The range of T_ℓ intersects each of the intervals $J_{n,0}$. Hence, if neither ℓ nor any $j > \ell$ is active for any n, then for every n some $j < \ell$ is active. Then in each step some integer is added to $L(n)$, while some smaller integers are removed. Eventually every integer less than ℓ is either in $L(n)$ or never active. (In some detail, if $\ell - 1$ is ever active, it will be put onto $L(n)$ and never removed, while if $\ell - 1$ is never active then that's fine too. Once $\ell - 1$ is settled, continue to the stage, if any, when $\ell - 2$ is active. Iterate. Since ℓ is finite, this eventually halts.) At that point there is nothing stopping ℓ from being active, which is what we wanted to show. This proves that f is not an elementary function.

We will now prove (3). By the Growth Theorem, it suffices to show the following claim:

> **(*)** Let $\ell \in \mathbb{N}$. Then there does not exist an elementary sequence $d_0 < d_1 < d_2 < \ldots$ such that for any k we have $f^\ell(d_m) > 2_k(d_m)$ for all but finitely many m.

Now, for every elementary sequence $d_0 < d_1 < d_2 < \ldots$, we have $T_i(\jmath) = d_\jmath$ for some i. Thus, by (*), it suffices to show the following claim:

> **(**)** Let $\ell \in \mathbb{N}$, and let T_i be total. Then there exists a k such that we have $f^{(\ell)}(T_i(m)) \leq 2_k(T_i(m))$ for infinitely many m.

The proof of (**) splits into two cases.

Case I: $T_i(m) \in J_{n,\jmath}$ with $\jmath > 0$ for infinitely many m. Then, for $n > i$ we have that such an interval $J_{n,\jmath}$ is not the last interval in the chain $J_{n,0}, \ldots, J_{n,k}$. Hence $f(x) = 2^x$ holds true on $[T_i(m), 2_n(T_i(m))]$, and for $n > \ell$ we have $f^{\ell}(T_i(m)) = 2_{\ell}(T_i(m))$.

Case II: not Case I. Then, $T_i(m) \in J_{n,0}$ for all but finitely many m. If i is active infinitely often, then for the witness $z = T_i(m)$ to this we have $f(T_i(m)) = 2_i(T_i(m))$, and $f^{(\ell)}(T_i(m)) = 2_{i+\ell-1}(T_i(m))$ for $i + \ell \leq n$, which suffices. If not, then i is active only finitely often. Once i is no longer active, it is never added to $L(n)$, but it is eventually removed from $L(n)$ (by the proof that f is not elementary). Once that happens, for each interval $J_{n,0}$ containing a value $T_i(m)$, i was not active because of some pair j, z with $2_j(z) \leq 2_i(T_i(m))$. But then again we have $f^{(\ell)}(T_i(m)) \leq 2_{i+\ell-1}(T_i(m))$ for $i + \ell \leq n$.

This completes the proof that no function in $\deg(f)$ is controllably reducible to a function in $\mathbf{0}$. □

Corollary 6.6 *(i) There exist degrees* $\mathbf{a}$ *and* $\mathbf{b}$ *such that* $\mathbf{a}$ *is not controllably irreducible to* $\mathbf{b}$ *and vice versa. (ii) Any countable partial ordering can be embedded in the degrees slightly above* $\mathbf{0}$.

Proof. By Theorem 6.5 and the Density-Splitting Theorem, we have a degree $\mathbf{a}$ slightly above $\mathbf{0}$ and two incomparable degrees $\mathbf{b}_1, \mathbf{b}_2$ such that $\mathbf{0} < \mathbf{b}_i < \mathbf{a}$ (for $i = 1, 2$). By Theorem 6.4, $\mathbf{b}_1$ will not be controllably irreducible to $\mathbf{b}_2$, and $\mathbf{b}_2$ will not be controllably irreducible to $\mathbf{b}_1$. This proves (i).

Furthermore, we know any countable partial ordering can be embedded between two degrees $\mathbf{a}$ and $\mathbf{b}$ whenever $\mathbf{a} < \mathbf{b}$. Thus, (ii) follows from Theorem 6.5 and Theorem 6.4. □

7 A Σ_1-complete first-order theory

In this section we give a first-order theory for deriving theorems on honest elementary degrees. We will prove that this theory is powerful enough to derive any true Σ_1-statement, that is, any true statement in the form $\exists x_1, \ldots, x_n A$ where A is a quantifier-free and does not contain other variables than $x_1, \ldots, x_n$. The reader should be aware that the proofs in this section may be a bit sketchy.

Definition 7.1 *Let*

$$a \cup b = c \quad \equiv \quad a \leq c \ \wedge \ b \leq c \ \wedge \ \forall d \left[a \leq d \ \wedge \ b \leq d \ \rightarrow \ c \leq d \right]$$

and let

$$a \cap b = c \quad \equiv \quad a \geq c \ \wedge \ b \geq c \ \wedge \ \forall d \left[a \geq d \ \wedge \ b \geq d \ \rightarrow \ c \geq d \right].$$

Furthermore, let $a \mid b \equiv a \not\leq b \wedge b \not\leq a$ *and* $a < b \equiv a \leq b \wedge a \neq b$.

Let $\mathcal{L}$ *be the first-order language* $\{\leq, \cdot', 0\}$*, and let* T *be an* $\mathcal{L}$*-theory which in addition standard axioms stating that* $\leq$ *is a partial ordering, contains the following axioms:*

- $\forall a \left[0 \leq a \right]$ *(Bottom Element)*
- $\forall a, b \left[a \leq b \rightarrow a' \leq b' \right]$ *(Monotonicity)*
- $\forall a \left[a \neq a' \right]$ *(Strictness)*
- $\forall a, b \exists c \left[a \cup b = c \right]$ *and* $\forall a, b \exists c \left[a \cap b = c \right]$ *(Lattice)*
- $\forall a, b, c \left[a \cup (b \cap c) = (a \cup b) \cap (a \cup b) \right]$ *(Distributivity)*
- $\forall a, b \left[a < b \ \rightarrow \ \exists c_1, c_2 \left[c_1 \mid c_2 \ \wedge \ c_1 \cap c_2 = a \ \wedge \ c_1 \cup c_2 = b \right] \right]$ *(Density)*
- $\forall a \exists b \left[a < b \ \wedge \ b' = a' \right]$ *(Low Degrees)*

- $\forall a \exists b\, [\, b < a' \;\wedge\; b' = a''\,]$ *(High Degrees)*
- $\forall a, b\, [\, a' \leq b \leq a'' \;\rightarrow\; \exists c\, [\, c \leq a \;\wedge\; c' = b\,]]$ *(Jump Inversion)*
- $\forall a, b, c\, [\, a < b \leq a' \;\wedge\; b \not\leq c \;\;\rightarrow\;\; \exists d\, [\, a < d < b \;\wedge\; d' = a' \;\wedge\; d \not\leq c\,]]$ *(Pendulum)*
- $\forall a, b[a' \cap b' = (a \cap b)']$.

□

Note that $\cap$ and $\cup$ are not symbols of the language $\mathcal{L}$, but all the axioms can be reduced to first-order statements over $\mathcal{L}$ in an obvious way. That $\cap$ distributes over $\cup$, that is $a \cap (b \cup c) = (a \cap b) \cup (a \cap b)$, follows from the axioms, see Birkhoff [**1**].

Definition 7.2 *A sublattice L of a jump lattice is* complete *when*

$$a, b \in L \;\wedge\; a' < b \;\;\Rightarrow\;\; a' \in L\,.$$

A lattice L is connected *if for any two elements $x, y \in L$ there exists a sequence of elements $z_1, \ldots, z_k \in L$ such that*

- *$x = z_1$ and $y = z_k$*
- *$z_i < z_{i+1}$ or $z_i > z_{i+1}$ (for $i \in \{1, \ldots, k-1\}$)*

□

Lemma 7.3 *Let L be a finite complete and connected sublattice of a jump lattice which is a model of T. There exists a homomorphism $L \rightarrow \mathbb{N}$ where the jump in $\mathbb{N}$ is the successor function.*

Proof. If L is a complete connected lattice containing n points, then we can enumerate the points of L as $\ell_1, \ldots, \ell_n$ such that $\{\ell_1, \ldots, \ell_k\}$ is a complete and connected lattice for all $k \leq n$: choose ℓ_1 arbitrarily, and choose jumps or jump inverses of existing elements whenever this is possible.

We prove the lemma by induction on the number of elements in L. Suppose that L is a complete sublattice together with a homomorphism $\varphi : L \rightarrow \mathbb{N}$, and let ℓ be some point not occurring in L. If ℓ is neither the jump of an element in L, nor is $\ell' \in L$, then we define $\varphi(\ell)$ to be the maximum of $\{\varphi(x) : x < \ell\}$. Thus we have $\varphi(\ell) \geq \varphi(x)$ for all $x < \ell$. Since $<$ is transitive, this also implies $\varphi(\ell) \leq \varphi(x)$ for all $x > \ell$.

If $\ell' \in L$, we put $\varphi(\ell) = \varphi(\ell') - 1$. If this happen to be negative, we just increase all values of φ by 1. As L is a complete lattice, there are no elements $x \in L$ with $x < \ell$. Suppose that $x > \ell$. Then $x' > \ell'$, hence $\varphi(x') \geq \varphi(\ell')$, and therefore $\varphi(x) \geq \varphi(\ell)$. A similar argument applies if there is some $x \in L$ with $x' = \ell$. □

Lemma 7.4 *Let $\mathfrak{L}$ be any model of T. Let L be a finite lattice, and let $a, b, c_1, \ldots, c_n$ elements of $\mathfrak{L}$ such that $a < b \leq a'$ and $b \mid c_i$ for $i = 1, \ldots, n$. Then, there exists an embedding $\psi : L \rightarrow \mathfrak{L}$ such that for any $x \in \psi(L)$ we have*

- *$a < x < b$*
- *$x \mid c_i$ for $i = 1, \ldots, n$*
- *$x' = a'$.*

Proof. To prove this lemma, we must use that $\mathfrak{L}$ satisfies the Pendulum Axiom and the Density Axiom. We omit the details. □

Lemma 7.5 *Let L be a finite jump lattice which is contained in a model of T, and let $\mathfrak{L}$ be an arbitrary model of T. Then there exists an embedding $\psi : L \rightarrow \mathfrak{L}$.*

Proof. We can w.l.o.g. assume that the lattice L is complete and connected. Let φ be the homomorphism given by Lemma 7.3, and assume that $n = \max\{\varphi(a) \mid a \in L\}$. Furthermore, let $L(k) = \{a \in L \mid \varphi(a) = k\}$. We will call $L(k)$ the k^{th} *level of* L. We can w.l.o.g. that there is only one element of level $L(n)$ and that each element of level k jumps to an element of level $k+1$, that is, for each $a \in L(k)$ there exists $b \in L(k+1)$ such that $a' = b$.

We will construct the embedding $\psi : L \to \mathfrak{L}$ level by level. First we construct $\psi : L(n) \to \mathfrak{L}$, then we construct $\psi : L(n-1) \to \mathfrak{L}$, and so on. There is only one degree a at level n, let $\psi(a)$ be an arbitrary degree strictly between $\mathbf{0}^{[n]}$ and $\mathbf{0}^{[n+1]}$.

Assume that we have constructed $\psi : L(k+1) \to \mathfrak{L}$. We will now construct $\psi : L(k) \to \mathfrak{L}$. Let $m_0, m_1, \ldots, m_{n_k}$ be an enumeration of the elements in $L(k+1)$ such that m_i is a maximal element in the set $\{m_i, \ldots, m_{n_k}\}$, and let

$$\mathrm{inv}(a) \;\; = \;\; \{\, b \mid b \in L(k) \;\wedge\; b' = a \,\} \; .$$

Now, $\mathrm{inv}(m_0), \mathrm{inv}(m_1), \ldots, \mathrm{inv}(m_{n_k})$ are disjunct sets, and

$$L(k) \;\; = \;\; \mathrm{inv}(m_0) \;\cup\; \mathrm{inv}(m_1) \;\cup\; \ldots \;\cup\; \mathrm{inv}(m_{n_k}) \; .$$

We construct the embedding $\psi : L(k) \to \mathfrak{L}$ by constructing the embedding $\psi : \mathrm{inv}(m_0) \to \mathfrak{L}$, then the embedding $\psi : \mathrm{inv}(m_1) \to \mathfrak{L}$, and so on.

Here is how to construct $\psi : \mathrm{inv}(m_i) \to \mathfrak{L}$ (for any $i \in \{0, \ldots, n_k\}$). Pick a maximal element $a \in \mathrm{inv}(m_i)$. The embedding ψ is now defined for all $b \in L$ such that $b > a$. Let $\alpha \in \mathcal{L}$ be given by $\alpha = \bigcap\{\psi(b) \mid b > a\}$. Now we have $\alpha' \geq \psi(a') \geq \alpha$ as $\mathcal{L}$ satisfies the axiom $\forall a, b[a' \cap b' = (a \cap b)']$. As $\mathfrak{L}$ satisfies the Jump Inversion Axiom, the Low Degree Axiom and the Pendulum Axiom, we can now find a suitable interval where we by Lemma 7.4 can embed all elements in $\mathrm{inv}(m_i)$ that cannot be distinguished from a by comparing them to elements already embedded. Next we consider a maximal element in $\mathrm{inv}(m_i)$ not yet treated, and construct ψ on the set of elements equivalent to this element as we did for the elements equivalent to a. Continuing downwards in this way we construct ψ for all elements in $\mathrm{inv}(m_i)$. $\square$

Theorem 7.6 (Σ_1-completeness) *Let $\mathfrak{H}$ denote the $\mathcal{L}$-structure of honest elementary degrees (our standard model for T), and let A be a Σ_1-statement in the language $\mathcal{L}$. Then*

$$\mathfrak{H} \models A \;\;\Leftrightarrow\;\; T \vdash A \; .$$

Proof. By Theorem 7.5, we know that if a finite jump lattice does not embed into an arbitrary model for T, the it will not embed into any model of T. Thus, a Σ_1-statement A will be satisfied in all models for T if, and only if, A is satisfied in some model for T. By the Completeness Theorem for first-order logic, we have

$$\mathfrak{H} \models A \;\;\Leftrightarrow\;\; T \models A \;\;\Leftrightarrow\;\; T \vdash A \; .$$

$\square$

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