MATH DAY 2015 at FAU

Competition A–Individual

SOLUTIONS

1. In a quiz of 20 problems, each correct answer is awarded 7 points, and each incorrect answer gets a deduction of 2 points. There is no award nor deduction for any questions omitted. Now Peter gets 87 points in the quiz. How many questions did he omit?

(A) 1 **(B)** 2 **(C)** 5 **(D)** 7 **(E)** 9

Solution. If Peter answers x questions correctly, y incorrectly, then $7x - 2y = 87$. There is a method to solve equations such as this one, but it is quite easy to guess that $x = 13, y = 2$ is a solution. It is also easy to see that it is the only solution with $x, y \geq 0$ and $x + y \leq 20$

The correct solution is **C.**

2. Your math teacher told you that in the year N^2, her age will be N. How old is she now in 2015?

(A) 28 **(B)** 36 **(C)** 40 **(D)** 42 **(E)** NA

Solution. We have $44^2 = 1936$, $45^2 = 2025$, $46^2 = 2116$. The only one that makes sense of these three is $45^2 = 2025$; so 10 years from now she will be 45 years old; thus she is 35 years old now.

The correct solution is **E.**

3. The numbers 1 to 20 are arranged in such a way that the sum of each pair of adjacent numbers is a prime:

$$20,\ p,\ 16,\ 15,\ 4,\ q,\ 12,\ r,\ 10,\ 7,\ 6,\ s,\ 2,\ 17,\ 14,\ 9,\ 8,\ 5,\ 18,\ t.$$

What is the number s?

(A) 1 **(B)** 3 **(C)** 11 **(D)** 13 **(E)** 19

Solution. s has to be 1 or 11. However the only choices for r and q are 1 and 19, so that one of r, q has to equal 1. Thus $s = 11$.

The correct solution is **C.**

4.⋆ By the divisors of an integer $n > 1$ we understand all **positive** integers, including n and 1, of which n is a multiple. Find the sum of all positive integer divisors of 2015. Write your answer directly onto the answer sheet.

Solution. **The hard way:** The divisors of 2015 are 1, 5, 13, 31, 65, 155, 403, 2015. Adding $1 + 5 + 13 + 31 + 65 + 155 + 403 + 2015 = 2688$. **The easier way:** Knowing a bit of number theory, if $\sigma(n)$ is the sum of the divisors of n one has that $\sigma(m \cdot n) = \sigma(n) \cdot \sigma(m)$ when n, m are relatively prime, and $\sigma(p^k) = (p^{k+1} - 1)/(p - 1)$, which is $p + 1$ if $k = 1$. Since $2015 = 51331$, the sum of its divisors is $(1 + 5)(1 + 13)(1 + 31) = 2688$.

The correct solution is **2688.**

5. By the divisors of an integer $n > 1$ we understand all **positive** integers, including n and 1, of which n is a multiple. How many integers $< 22,500$ have exactly 15 divisors?

(A) 3 **(B)** 15 **(C)** 21 **(D)** 35 **(E)** 45

Solution. If $n > 1$ we can write it in the form $n = p_1^{e_1} \cdots p_r^{e_r}$, where $p_1, \dots, p_n$ are distinct primes and $e_1, \dots, e_r$ are positive integers. It follows that the number of divisors of n is $(e_1 + 1) \cdots (e_r + 1)$. If this product equals 15 the possibilities are: $r = 1$ and $e_1 = 14$, or $r = 2$, and $e_1 = 3, e_2 = 5$. Thus either $n = p^{14}$ for some prime p, or $n = p^2 q^4$ for distinct primes p, q. If $p^{14} < 22,500 = 150^2$ we get $p^7 < 150$. Now $2^7 = 128 < 150$, but $3^5 = 243$ is already > 150. Thus the only choice for $n = p^{14}$ is with $p = 2$.

Let's count now how many choices there are for p, q with $p^2 q^4 < 150^2$. Then $q^2 < 150$ and this shows that q must be a prime < 13, so $q = 2, 3, 5, 7$ or 11. Also $pq^2 < 150$.

If $q = 2$, then $4p < 150$ implies $p < 150/4 = 37.5$, thus p is one of $3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37$; 11 choices in all.

If $q = 3$, then $p < 150/9 = 16.6\ldots$, so $p = 2, 5, 7, 11, 13$; 5 choices in all.

If $q = 5$, then $p < 150/25 = 6$, so $p = 2, 3$; 2 choices in all.

If $q = 7$, then $p < 150/49 = 3...$: so $p = 2, 3$, also 2 choices.

If $q = 11$, then $p < 150/121 < 2$; there is no such p.

The total number of choices are $1 + 11 + 5 + 2 + 2 = 21$.

The correct solution is **C.**

6. Find the number of trailing 0's of $1000! = 1 \cdot 2 \cdot 3 \cdots 999 \cdot 1000$. By "trailing 0's" we understand the 0's that follow the last non zero digit of the number; for example $7! = 5040$ has one trailing 0, $12! = 479001600$ has two trailing 0's.

(A) 178 **(B)** 249 **(C)** 375 **(D)** 415 **(E)** NA

Solution. There being considerably more even numbers than multiples of 5, the number of zeros equals the number of factors of 5 that one can find in the range 1–1000. Every multiple of 5 that is not a multiple of 25 contibutes a zero, multiles of 25 that are not multiples of 125 contribute 2, and so forth.

$1000/5 = 200$ multiples of 5, contributing one 0.

Of these, $1000/25 = 40$ are multiples of 25, contributing an extra 0.

Of the multiples of 25, $1000/125 = 8$ re multiples of 125, contributing another 0.

There is one multiple of 625.

The total number of zeros is thus $200 + 40 + 8 + 1 = 249$

The correct solution is **B.**

7. If $a^2 - a - 10 = 0$, then $(a + 1)(a + 2)(a - 4)$ is

(A) an integer. **(B)** positive and irrational. **(C)** negative and irrational.
(D) rational but not an integer. **(E)** NA

Solution. $(a+1)(a+2)(a-4) = a^3 - a^2 - 10a - 8 = a(a^2 - a - 10) - 8 = -8$.

The correct solution is **A.**

8. Let a, b be real numbers such that $(a + bi)^2 = 9 + 40i$, where $i = \sqrt{-1}$ is the imaginary unit. Assuming $a > 0$, a equals

(A) 3 **(B)** 5 **(C)** 8 **(D)** 20 **(E)** NA

Solution. $(a + bi)^2 = (a^2 - b^2) + 2iab$, so $a^2 - b^2 = 9$, $2ab = 40$. Substituting $b = 20/a$ into the first equation, we get after multiplying by a^2 and rearranging, $a^4 - 9a^2 - 400 = 0$. Solving this quadratic equation for a^2 we get $a^2 = 25$ (we discard the negative solution), thus $a = 5$.

The correct solution is **B.**

9. Find the sum of all the roots, real **and** complex, of the equation $x^{2015} + (x - \dfrac{1}{4})^{2015} = 0$, given that all roots are distinct.

$$\textbf{(A)}\ \frac{2015}{8} \qquad \textbf{(B)}\ \frac{2015}{4} \qquad \textbf{(C)}\ \frac{2015}{2} \qquad \textbf{(D)}\ 2015 \qquad \textbf{(E)}\ 2015^2$$

Solution. If we expand the expression, the two first terms will be $2x^{2015} - \frac{2015}{4}x^{2014} + \cdots$. By Viète's relations, the summ of the roots equals $-(-2015/4)/2$.

The correct solution is **A.**

10. Assume a, b are real numbers larger than 1 and $\log_a b = 2 \log_b a$. Then $\log_a b$ equals

$$\textbf{(A)}\ a \qquad \textbf{(B)}\ b \qquad \textbf{(C)}\ e \text{ (the base of natural logarithms)} \qquad \textbf{(D)}\ \sqrt{2} \qquad \textbf{(E)}\ 1$$

Solution. If $x = \log_a b = 2 \log_b a$ then $a^x = b$. Taking log in base b, $1 = x \log_b a = \frac{2}{x^2}$.

The correct solution is **D.**

11.* Find the sum of all distinct 5 digits numbers that contain only the digits 1, 2, 3, 4, 5, and 6, each at most once. So, for example, some of the numbers to be added up are 12345, 34652, 31246. On the other hand, a number such as 33456 is not to be considered. Write the answer directly on the answer sheet.

Solution. If $n = d_1d_2d_3d_4d_5$ is a five digits number we say d_5 is in the units' place, d_4 in the tens' place, d_3 in the hundreds' place, d_2 in the thousands' place and d_1 in the ten thousands' place. Concentrate on one of these places, say the units' place. The digit 1 will appear in that place as many times as the digits 2, 3, 4, 5, 6 in the other 4 places, which is $5 \cdot 4 \cdot 3 \cdot 2 = 120$ times. The same goes for the other choices. Adding digits over any of the places will thus result in

$$120(1 + 2 + 3 + 4 + 5 + 6) = 2520.$$

The total sum will thus be $2520 \times 10^4 + 2520 \times 10^3 + 2520 \times 10^2 + 2520 \times 10 + 2520 = 27999720$.

The correct solution is **27999720.**

12. Find the remainder when $x^{2015} + x^{2014} + x^{2013} + x^2(x + 1)$ is divided by $x^3 - 1$.

$$\textbf{(A)}\ 2 \qquad \textbf{(B)}\ 2x + 1 \qquad \textbf{(C)}\ x^2 + x + 1 \qquad \textbf{(D)}\ x^2 + 2x + 2 \qquad \textbf{(E)}\ 2x^2 + x + 2$$

Solution. Recall that if $p(x)$ is a polynomial, then the remainder of dividing $p(x)$ by $x - a$ is $p(a)$; that is

$$p(x) = (x - a)q(x) + p(a)$$

for some polynomial $q(x)$. Thus

$$x^{2013} = (x - 1)q(x) + 1^{2013} = (x - 1)q(x) + 1.$$

Dividing $x^2(x + 1)$ by $x^3 - 1$ we get $x^2(x + 1) = x^3 + x^2 = (x^3 - 1) + x^2 + 1$. Thus

$$\begin{aligned} x^{2015} + x^{2014} + x^{2013} + x^2(x + 1) &= (x^2 + x + 1)x^{2013} + x^2(x + 1) \\ &= (x^2 + x + 1)\left((x - 1)x^{2013}q(x) + 1\right) + (x^3 - 1) + x^2 + 1 \\ &= (x - 1)^3\left(x^{2013}q(x) + 2\right) + 2x^2 + x + 2 \end{aligned}$$

The remainder is $2x^2 + x + 2$.

The correct solution is **E.**

13.* Let x, y, z be real numbers such that

$$\begin{array}{ccccccc} x & + & y & + & z & = & 17 \\ x^2 & + & y^2 & + & z^2 & = & 123 \\ x^3 & + & y^3 & + & z^3 & = & 953 \end{array}$$

Find xyz. Write your answer directly onto the answer sheet.

Solution. We have

$$\begin{aligned} (x+y+z)^3 &= x^3+y^3+z^3+3x^2y+3x^2z+3xy^2+3y^2z+3xz^2+3yz^2+6xyz \\ &= x^3+y^3+z^3+3x^2(y+z)+3y^2(x+z)+3z^2(x+y)+6xyz \\ &= x^3+y^3+z^3+3x^2(x+y+z)-3x^3+3y^2(x+y+z)-3y^3+3z^2(x+y+z)-3z^3+6xyz \\ &= -2\left(x^3+y^3+z^3\right)+3\left(x^2+y^2+z^2\right)(x+y+z)+6xyz. \end{aligned}$$

Thus.

$$xyz = \frac{1}{6}\left((x+y+z)^3+2\left(x^3+y^3+z^3\right)-3\left(x^2+y^2+z^2\right)(x+y+z)\right).$$

In our case $xyz = \dfrac{17^3+2\cdot 953-3\cdot 123\cdot 17}{6} = 91.$

The correct solution is **91.**

14.* The legs a, b of a right triangle ABC (with right angle at C) satisfy the equation $a^2-4ab+b^2=0$. Let s be the sine of the smaller of the two acute angles of the triangle. Then $s^2 = \dfrac{m-\sqrt{n}}{k}$; where m, n, k are positive integers and k is square free. Find m, n, k.

Write your answer directly onto the answer sheet.

Solution. Suppose the smaller angle is the angle at A so that $a \le b$. Setting $x = a/b$ and dividing the equation for a, b by b^2, we get that $x^2-4x+1=0$, thus $x = (4\pm 2\sqrt{3})/2 = 2\pm\sqrt{3}$. Now $x = a/b$ is the tangent of A, so we have to choose the smaller of the two solutions x. We have that $\tan A = 2-\sqrt{3}$ while $2+\sqrt{3} = tanB = \cot A$. Then

$$\frac{2-\sqrt{3}}{2+\sqrt{3}} = \frac{\tan A}{\cot A} = \frac{\sin^2 A}{\cos^2 A} = \frac{s^2}{1-s^2}.$$

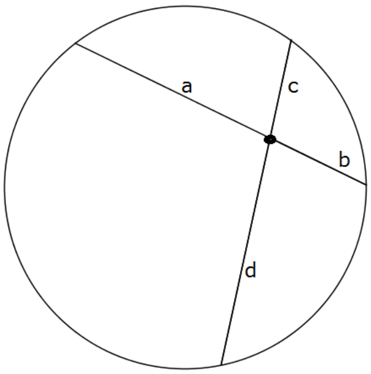
Solving, $s^2 = (2-\sqrt{3})/4$.

The correct solution is $m=2, n=3, k=4$**..**

15. Assume the graph of $y = x^2+px+q$ intersects the coordinate axes at three points. The circle through these three points passes through a fixed point in the coordinate plane; a point that does not depend on the values of p or q. What are the coordinates of this fixed point as p and q vary?

(A) $(0,0)$ **(B)** $(0,-1)$ **(C)** $(0,1)$ **(D)** $(1,1)$ **(E)** NA

Solution. The *intersecting chords theorem* states that if two chords of a circle intersect, the product of the lengths of the two segments into which each chord breaks up are equal.

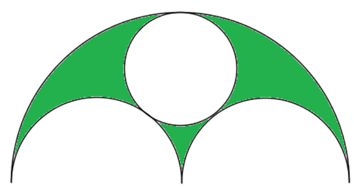


The points at which the given graph intersects the axes have coordinates $(a, 0), (b, 0)$, where a, b are the roots of the equation $x^2 + px + q = 0$, and $(0, q)$. A circle through these points will intersect the y-axis at a fourth point $(0, c)$. Assuming first $q > 0$, then $\alpha > 0, \beta > 0$ and by the intersecting chord theorem, $cq = ab$; since $ab = q$ we get $c = 1$. The point is $(0, 1)$. The same result holds if $q \leq 0$; in which case α, β have opposite parity.

The correct solution is **C.**

16.

In the figure to the right, the circle and semicircles are tangent to each other as indicated. The two smaller semicircles have equal radii. Let A be the area of the shaded region and let B be the area of the circle (there is only one full circle in the figure). Find the ratio A/B.



(A) $4 : 3$ **(B)** $5 : 4$ **(C)** $1 : 1$ **(D)** $4 : 5$ **(E)** $3 : 4$

Solution. Let r be the radius of the smaller semicircles. Then the radius of the large semicircle is $2r$. If ρ is the radius of the circle, then $2r - \rho$ is the length of the one of the legs of the right triangle marked in the figure below; the other leg has length r while the hypotenuse has length $r + \rho$.

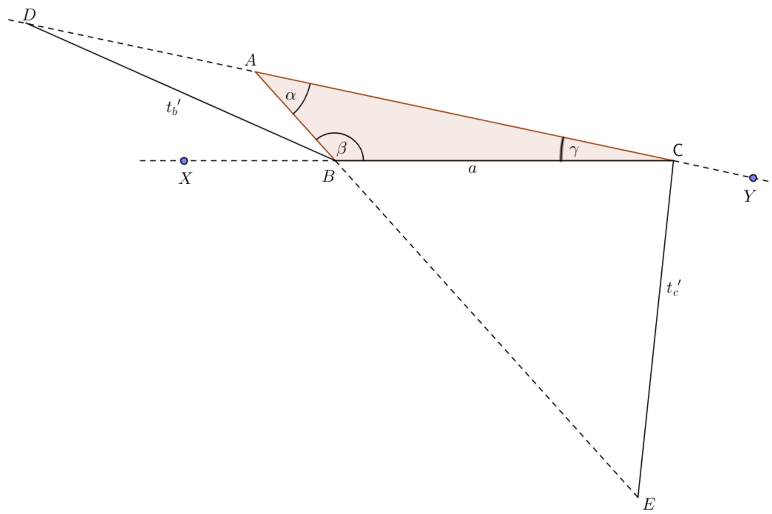
By the Theorem of Pythagoras, $(r + \rho)^2 = r^2 + (2r - \rho)^2$ from which $\rho = 2r/3$. We can now find all areas in terms of r: The shaded area is

$$A = 2\pi r^2 - \pi r^2 - \pi \left(\frac{2r}{3}\right)^2 = \frac{5}{9}\pi r^2;$$

Area of the circle: $\pi \left(\frac{2r}{3}\right)^2 = \frac{4}{9}\pi r^2$, thus $A/B = 5/4$.

The correct solution is **B.**

17.* ABC is a triangle with both external angle bisectors t'_a and t'_b equal to a. That is BD bisects angle $\angle XBA$, EC bisects angle $\angle BCY$ and $|BD| = |BC| = |CE|$. Calculate, **in degrees** angles α, β, and γ.



Write your answer in the appropriate place on the answer sheet.

Solution. This is an "angle chasing" exercise. Triangle BCD is isosceles, thus $\angle ADB = \angle ACB = 180 - (\alpha + \beta)$. Now, adding the angles in triangle ABD,

$$180 = \angle ADB + \angle DBA + \angle BAD = (180 - \alpha - \beta) + \frac{1}{2}(180 - \beta) + (180 - \alpha).$$

Simplifying, we get $4\alpha + 3\beta = 540$. Adding angles in triangle BCE, which is also isosceles with $\angle BEC = \angle CBE = 180 - \beta$,

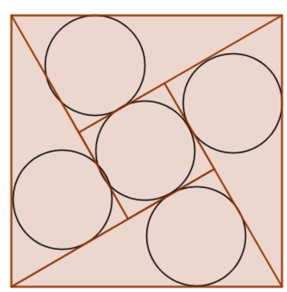
$$180 = \angle CBE + \angle BEC + \angle CBE = 2(180 - \beta) + \frac{1}{2}(\alpha + \beta).$$

This simplifies to $-\alpha + 3\beta = 360$. Solving the two equations for α, β gives $\alpha = 36, \beta = 132$, thus $\gamma = 12$.

The correct solution is $\alpha = 36°$**,** $\beta = 132°$**,** $\gamma = 12°$**..**

18.

A square of sides of length 2 has been partitioned into five regions, four congruent triangles and a square as shown in the picture on the right. The inscribed circles are all equal. If r is the radius of these circles then r equals



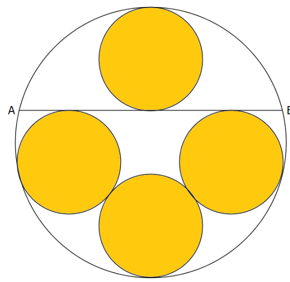
(A) $\frac{\sqrt{3}-1}{2}$ **(B)** $\frac{\sqrt{3}+1}{8}$ **(C)** $\frac{2\sqrt{3}-1}{8}$ **(D)** $\frac{2\sqrt{3}+1}{16}$ **(E)** NA

Solution. Because the central figure is a square, the four triangles must be right triangles. It helps to know that if a circle of radius r is inscribed in a right triangle of legs a, b and hypotenuse c, then $a + b = c + 2r$. We calculate the area of the big square in two ways, assuming a, b are the legs (say $b \geq a$) and c the hypotenuse of the congruent triangles; c is also the side of the square. The area of the square is thus c^2. But it also equals four times the area of the triangles (so $4 \times \frac{1}{2}ab = 2ab$) plus the area of the central square, which is $(2r) \times (2r) = 4r^2$. Thus $c^2 = 2ab + 4r^2$. By Pythagoras, $c^2 = a^2 + b^2$, thus $a^2 + b^2 = 2ab + 4r^2$, hence $(b - a)^2 = 4r^2$. Taking square roots we get $r = (b - a)/2$. Returning to the equality $a + b = c + 2r$, we can now write it in the form $a + b = c + b - a$, from which $a = c/2$. Then $b = \sqrt{c^2 - a^2} = c\sqrt{3}/2$ and $r = (b - a)/2 = (\sqrt{3} - 1)c/4$. Setting $c = 2$, we see that $r = (\sqrt{3} - 1)/2$.

The correct solution is **A.**

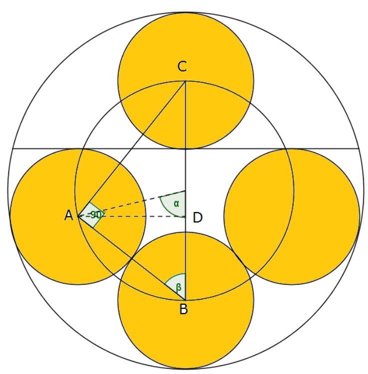
19.

Four circles of radius r are included and are tangent to a circle of radius R; one of these circles is above, and tangent to the secant AB, the other three circles are below this secant; two of them tangent to the secant and to the third circle, as shown in the picture on the right. The ratio r/R equals



(A) $\dfrac{3+\sqrt{5}}{16}$ **(B)** $\dfrac{\sqrt{5}-1}{2}$ **(C)** $\dfrac{\sqrt{5}-1}{4}$ **(D)** $\dfrac{2\sqrt{5}-3}{4}$ **(E)** $\dfrac{3-\sqrt{5}}{2}$

Solution. The centers of the four little circles lie on a circle, concentric with the big one, of radius $R-r$. The triangle ABC shown in the picture below is a right triangle, with right angle at A, since that angle subtends the diameter of the circle of radius $R-r$. In the picture, AD is perpendicular to BC.

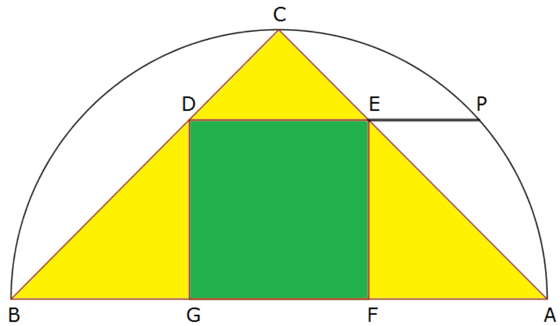


The angle at C of the triangle ABC is inscribed to the circle of radius $R-r$, subtending the same secant AB as the central angle α, thus the angle at C equals $\alpha/2$ and $\beta = \pi/2 - \angle C = \pi/2 - \alpha/2$. Since α is a central angle of a circle of radius $R-r$, subtending a secant of length $2r$, one sees that $\sin(\alpha/2) = r/(R-r)$. Thus $\cos\beta = \sin(\alpha/2) = r/(R-r)$ and $|BD| = 2r\cos\beta = 2r^2/(R-r)$. The diameter of the big circle is $2R$; we can also see that it equals $|BD| + 4r = \dfrac{2r^2}{R-r} + 4r$. Equating this last quantity to $2R$ and rearranging we get the equation $r^2 - 3Rr + R^2 = 0$, which has the solutions $r = [(3 \pm \sqrt{5})/2]R$. Ignoring the solution that is larger than R, we get $r = [(3-\sqrt{5})/2]R$. Notice that this shows that the secant, AB in the first picture, cuts the vertical diameter in golden ratio. In fact, the ratio is $2R/(2R-2r) = 2/(\sqrt{5}-1) = (\sqrt{5}+1)/2$.

The correct solution is **E.**

20. $DEFG$ is a square inscribed in an isosceles right triangle ABC. The side DE is extended to intersect the circumcircle of the triangle at P. The ratio $DE : EP$ is equal to

(A) $2:1$ **(B)** $3:2$ **(C)** $5:4$ **(D)** $\sqrt{2}:1$ **(E)** NA



Solution. If the semicircle has radius r, then the triangle has bas of length $2r$. the square sides of length $2r/3$. In the picture below we added the midpoint R of the segment DE, the center O of the semicircle, the segments OR and OP. We have

$$|RP| = \sqrt{|OP|^2 - |OR|^2} = \sqrt{r^2 - (2r/3)^2} = \frac{r\sqrt{5}}{3}.$$

Thus

$$\frac{|DE|}{|EP|} = \frac{|DE|}{|RP| - (|DE|/2)} = \frac{\frac{2r}{3}}{\frac{r\sqrt{5}}{3} - \frac{r}{3}} = \frac{2}{\sqrt{5}-1} = \frac{1+\sqrt{5}}{2},$$

which happens to be the golden mean.

The correct solution is **E.**