

MATH DAY 2015 at FAU
Competition A–Individual

NOTE:

1. Enter your name on the answer sheet. Detach the answer sheet from the rest of the test before handing it in. You may keep the test as such.
2. **Starred Problems** Fifteen of the problems are multiple choice. For the other five problems (identified with a star beside their number) the answer is **in every case** a positive integer or, in two cases, three positive integers, which you enter directly beside the problem number on the answer sheet. **Make sure you write clearly.**
3. In the multiple choice questions, the option NA stands for “None of the previous answers is correct.”
4. The notation AB is used to indicate the line through the points A, B , or the segment from A to B ; $|AB|$ denotes the length of the segment AB .
5. $\log_b a$ denotes the logarithm in base b of a ; $\log_b a = c$ if and only if $b^c = a$.
6. As a symbol, i denotes the imaginary unit; $i^2 = -1$.
7. If n is a non-negative integer, then $n!$ stands for the product of all positive integers in the range 1 to n if $n \geq 1$, with $0!$ defined to be 1. That is:

$$0! = 1, 1! = 1, 2! = 2, 3! = 2 \cdot 3 = 6, 4! = 2 \cdot 3 \cdot 4 = 24, 5! = 2 \cdot 3 \cdot 4 \cdot 5 = 120, \text{ etc.}$$

8. Do NOT assume that pictures are drawn to scale. They are merely intended as a guide.
9. The problems are not ordered by degree of difficulty. Problem n could be harder than problem $n + 1$.

THE QUESTIONS

1. In a quiz of 20 problems, each correct answer is awarded 7 points, and each incorrect answer gets a deduction of 2 points. There is no award nor deduction for any questions omitted. Now Peter gets 87 points in the quiz. How many questions did he omit?

(A) 1 (B) 2 (C) 5 (D) 7 (E) 9

2. Your math teacher told you that in the year N^2 , her age will be N . How old is she now in 2015?

(A) 28 (B) 36 (C) 40 (D) 42 (E) NA

3. The numbers 1 to 20 are arranged in such a way that the sum of each pair of adjacent numbers is a prime:

$$20, p, 16, 15, 4, q, 12, r, 10, 7, 6, s, 2, 17, 14, 9, 8, 5, 18, t.$$

What is the number s ?

(A) 1 (B) 3 (C) 11 (D) 13 (E) 19

- 4.* By the divisors of an integer $n > 1$ we understand all **positive** integers, including n and 1, of which n is a multiple. Find the sum of all positive integer divisors of 2015. Write your answer directly onto the answer sheet.

5. By the divisors of an integer $n > 1$ we understand all **positive** integers, including n and 1, of which n is a multiple. How many integers $< 22,500$ have exactly 15 divisors?

(A) 3 (B) 15 (C) 21 (D) 35 (E) 45

6. Find the number of trailing 0's of $1000! = 1 \cdot 2 \cdot 3 \cdots 999 \cdot 1000$. By "trailing 0's" we understand the 0's that follow the last non zero digit of the number; for example $7! = 5040$ has one trailing 0, $12! = 479001600$ has two trailing 0's.

(A) 178 (B) 249 (C) 375 (D) 415 (E) NA

7. If $a^2 - a - 10 = 0$, then $(a + 1)(a + 2)(a - 4)$ is

(A) an integer. (B) positive and irrational. (C) negative and irrational.
(D) rational but not an integer. (E) NA

8. Let a, b be real numbers such that $(a + bi)^2 = 9 + 40i$, where $i = \sqrt{-1}$ is the imaginary unit. Assuming $a > 0$, a equals

(A) 3 (B) 5 (C) 8 (D) 20 (E) NA

9. Find the sum of all the roots, real **and** complex, of the equation $x^{2015} + (x - \frac{1}{4})^{2015} = 0$, given that all roots are distinct.

(A) $\frac{2015}{8}$ (B) $\frac{2015}{4}$ (C) $\frac{2015}{2}$ (D) 2015 (E) 2015^2

10. Assume a, b are real numbers larger than 1 and $\log_a b = 2 \log_b a$. Then $\log_a b$ equals

(A) a (B) b (C) e (the base of natural logarithms) (D) $\sqrt{2}$ (E) 1

- 11.* Find the sum of all distinct 5 digits numbers that contain only the digits 1, 2, 3, 4, 5, and 6, each at most once. So, for example, some of the numbers to be added up are 12345, 34652, 31246. On the other hand, a number such as 33456 is not to be considered. Write the answer directly on the answer sheet.

12. Find the remainder when $x^{2015} + x^{2014} + x^{2013} + x^2(x + 1)$ is divided by $x^3 - 1$.

(A) 2 (B) $2x + 1$ (C) $x^2 + x + 1$ (D) $x^2 + 2x + 2$ (E) $2x^2 + x + 2$

- 13.* Let x, y, z be real numbers such that

$$\begin{aligned} x + y + z &= 17 \\ x^2 + y^2 + z^2 &= 123 \\ x^3 + y^3 + z^3 &= 953 \end{aligned}$$

Find the product xyz of x times y times z . Write your answer directly onto the answer sheet.

- 14.* The legs a, b of a right triangle ABC (with right angle at C) satisfy the equation $a^2 - 4ab + b^2 = 0$. Let s be the sine of the smaller of the two acute angles of the triangle. Then $s^2 = \frac{m - \sqrt{n}}{k}$; where m, n, k are positive integers and k is square free. Find m, n, k .

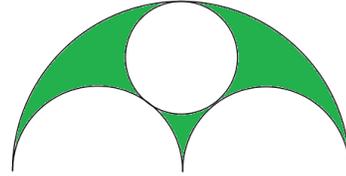
Write your answer directly onto the answer sheet.

15. Assume the graph of $y = x^2 + px + q$ intersects the coordinate axes at three points. The circle through these three points passes through a fixed point in the coordinate plane; a point that does not depend on the values of p or q . What are the coordinates of this fixed point as p and q vary?

(A) (0, 0) (B) (0, -1) (C) (0, 1) (D) (1, 1) (E) NA

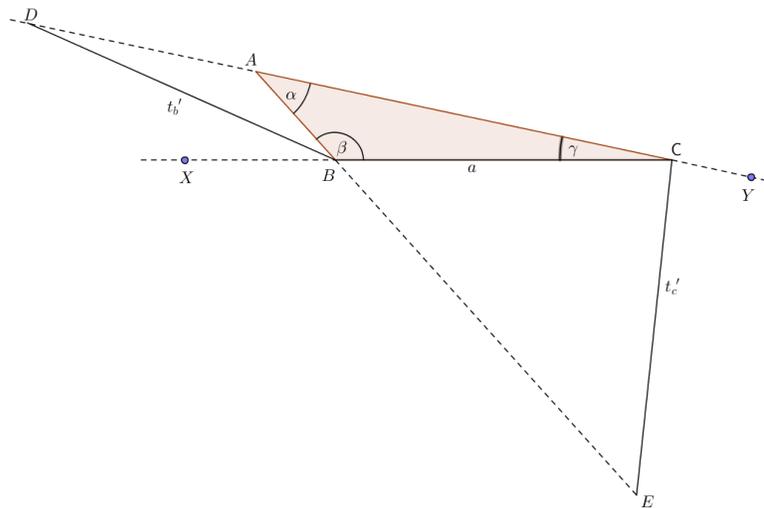
16.

In the figure to the right, the circle and semicircles are tangent to each other as indicated. The two smaller semicircles have equal radii. Let A be the area of the shaded region and let B be the area of the circle (there is only one full circle in the figure). Find the ratio A/B .



- (A) 4 : 3 (B) 5 : 4 (C) 1 : 1 (D) 4 : 5 (E) 3 : 4

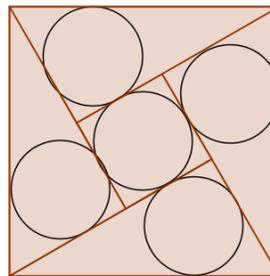
17.* ABC is a triangle with both external angle bisectors t'_a and t'_b equal to a . That is, BD bisects angle $\angle XBA$, EC bisects angle $\angle BCY$ and $|BD| = |BC| = |CE|$. Calculate, **in degrees** angles α , β , and γ .



Write your answer in the appropriate place on the answer sheet.

18.

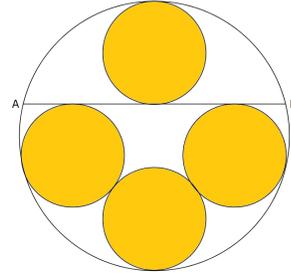
A square of sides of length 2 has been partitioned into five regions, four congruent triangles and a square as shown in the picture on the right. The inscribed circles are all equal. If r is the radius of these circles then r equals



- (A) $\frac{\sqrt{3}-1}{2}$ (B) $\frac{\sqrt{3}+1}{8}$ (C) $\frac{2\sqrt{3}-1}{8}$ (D) $\frac{2\sqrt{3}+1}{16}$ (E) NA

19.

Four circles of radius r are included in and are tangent to a circle of radius R ; one of these circles is above, and tangent to the secant AB , the other three circles are below this secant; two of them are tangent to the secant and to the third circle, as shown in the picture on the right. The ratio r/R equals



- (A) $\frac{3 + \sqrt{5}}{16}$ (B) $\frac{\sqrt{5} - 1}{2}$ (C) $\frac{\sqrt{5} - 1}{4}$ (D) $\frac{2\sqrt{5} - 3}{4}$ (E) $\frac{3 - \sqrt{5}}{2}$

20. $DEFG$ is a square inscribed in an isosceles right triangle ABC . The side DE is extended to intersect the circumcircle of the triangle at P . The ratio $DE : EP$ is equal to

- (A) 2 : 1 (B) 3 : 2 (C) 5 : 4 (D) $\sqrt{2} : 1$ (E) NA

