MATH DAY 2014 at FAU

Competition A–Individual

QUESTIONS AND SOLUTIONS

THE QUESTIONS

1. There is a written test that consists of 30 multiple-choice questions. A student will receive 5 points for each correct answer, 1 point for each answer left blank, and 0 point for each incorrect answer. Suppose that at the end of the written test, five students make the following statements:

   Alice says, "My test score is 147."
   Bob says, "My test score is 144."
   Cathy says, "My test score is 143."
   David says, "My test score is 141."
   Erica says, "My test score is 139."

   Only one of the five students could possibly be correct. Which one?

   (A) Alice **(B)** Bob **(C)** Cathy **(D)** David **(E)** Erica

   Be sure to circle the appropriate choice on the answer sheet!

   Solution. David could be right; if he got 28 answers right, one wrong and left one blank, his score would be $5 \cdot 28 + 1 = 141$. So David is the answer, assming the problem correctly posed. To see that it is, notice that if a student got a answers right, left b blank, and c wrong, then $a + b + c = 30$ and his/her score is $5a + b$. All alleged scores $5a + b \geq 139$; using $a = 30 - b - c$ we get $4b + 5c \leq 11$. This implies $b + c$ is at most 2; in particular $b \leq 2$. Now $5a + b = 147$ requires $147 - b$ a multiple of 5; given $b \leq 2$ it forces $b = 2$. Then $5a = 145$ so $a = 29$, impossible since there are only 30 questions. A score of 144 would require $b = 4$, not possible. Similarly 143 is not possible. Neither is 139, here b has to be 4.

   The correct solution is **D.**

2. Let $3^x = 4$, $4^y = 5$, $5^z = 6$, $6^u = 7$, $7^v = 8$, and $8^w = 9$. Find the value of the product $xyzuvw$.

   (A) 1 **(B)** 2 **(C)** $\sqrt{6}$ **(D)** 3 **(E)** 4

   Be sure to circle the appropriate choice on the answer sheet!

   Solution.
   $$9 = 8^w = 7^{vw} = 6^{uvw} = 5^{zuvw} = 4^{yzuvw} = 3^{xyzuvw}$$
   so that $xyzuvw = 2$.

   The correct solution is **B.**

3. What is the last digit d of the 9-digit number $19700019d$, given that the number is a prime?

   (A) 1 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** 9

   Be sure to circle the appropriate choice on the answer sheet!

Solution. Since $1+9+7+1+9=27$ is divisible by 9, we can rule out at once 3 and 9; otherwise the number would be divisible by 3 or 9, respectively. We can rule out all the even digits; or the number would be divisible by 2. We can also rule out 5; otherwise the number would be a multiple of 5. That leaves 7 and 1. If $d=7$, then

$$19700019d = 197000197 = 197 \times 1000001$$

is not prime. That leaves 1 as the only possibility,

The correct solution is **A.**

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4. Three **different** nonzero numbers a, b, and c are chosen so that

$$\frac{a+b}{c} = \frac{b+c}{a} = \frac{c+a}{b}.$$

What is the common value of these three quotients?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Be sure to circle the appropriate choice on the answer sheet!

Solution. Multiplying $\frac{a+b}{c} = \frac{b+c}{a}$ by ac and $\frac{b+c}{a} = \frac{c+a}{b}$ by ab we get

$$a^2+ab = bc+c^2 \qquad \text{and} \quad b^2+bc = ac+a^2,$$

from which

$$a(b-c) = c^2-b^2 = -(b-c)(b+c).$$

Since a,b,c are different, we have $a=-(b+c)$ or $a+b+c=0$. Assume now that a,b,c are any three numbers non-zero numbers such that $a+b+c=0$. Then

$$\frac{a+b}{c} = \frac{-c}{c} = -1, \quad \frac{b+c}{a} = \frac{-a}{a} = -1, \quad \frac{c+a}{b} = \frac{-b}{b} = -1.$$

The correct solution is **B.**

5. What is the remainder when $x^{2014} - x^{2013} + (x+1)^2$ is divided by x^2-1?

(A) $2x$ **(B)** $x-3$ **(C)** $2x+2$ **(D)** $x+3$ **(E)** 2

Be sure to circle the appropriate choice on the answer sheet!

Solution 1. Recall that if $p(x)$ is a polynomial, then the remainder of dividing $p(x)$ by $x-a$ is $p(a)$; that is

$$p(x) = (x-a)q(x) + p(a)$$

for some polynomial $q(x)$. Thus

$$x^{2013} = (x+1)q(x) + (-1)^{2013} = (x+1)q(x) - 1, \quad x+1 = (x-1)+2,$$

hence

$$x^{2014}-x^{2013}+(x+1)^2 = x^{2013}(x-1)+(x+1)(x+1) = (x-1)\left((x+1)q(x)-1\right)+(x+1)\left((x-1)+2\right) = (x^2-1)\left(q(x)+1\right)+(x+3)$$

The remainder is $x+3$

Solution 2. Since the divisor is of degree 2, the remainder is at most of degree 1, hence of the form $r(x) = ax+b$. With $p(x) = x^{2014} - x^{2013} + (x+1)^2$ the given polynomial, we have $p(x = q(x)(x^2-1) + r(x)$ so that $r(\pm 1) = p(\pm 1)$. This works out to

$$a+b = p(1) = 4, \quad -a+b = p(-1) = 2,$$

from which one gets at once $b=3$ and $a=1$.

The correct solution is **D.**

6. Suppose that α is a root of $x^4 + x^2 - 1$. What is the value of $\alpha^6 + 2\alpha^4$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5

Be sure to circle the appropriate choice on the answer sheet!

Solution. The roots of the equation satisfy

$$x^2 = \frac{-1 \pm \sqrt{5}}{2}.$$

If $\alpha^2 = (-1+\sqrt{5})2 = (\sqrt{5}-1)/2$,

$$\alpha^6 + 2\alpha^4 = \frac{5\sqrt{5} - 15 + 3\sqrt{5} - 1}{8} + 2\frac{5 - 2\sqrt{5} + 1}{4} = \sqrt{5} - 2 + 3 - \sqrt{5} = 1,$$

if $\alpha^2 = (-1-\sqrt{5})2 = -(\sqrt{5}-1)/2$,

$$\alpha^6 + 2\alpha^4 = -\frac{5\sqrt{5} + 15 + 3\sqrt{5} + 1}{8} + 2\frac{5 + 2\sqrt{5} + 1}{4} = -\sqrt{5} - 2 + 3 + \sqrt{5} = 1,$$

The correct solution is **A.**

7.* What is the coefficient of x^6 in the expansion of

$$(x+2)^{11} + (x+2)^{10}x + (x+2)^9x^2 + (x+2)^8x^3 + \cdots (x+2)x^{10} + x^{11}?$$

Write your answer directly on the answer sheet.

Solution.

$$(x+2)^{11} + (x+2)^{10}x + (x+2)^9x^2 + \cdots (x+2)x^{10} + x^{11} = \frac{(x+2)^12 - x^{12}}{(x+2) - x} = \frac{1}{2}\sum_{j=0}^{11} \binom{12}{j} 2^{12-j}x^j.$$

For $j = 6$ we get that the coefficient is

$$\frac{1}{2}\binom{12}{6} 2^6 = 32 \cdot \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} = 29568$$

The correct solution is **29568.**

8. How many positive real numbers x are solutions to the equation below?

$$\sqrt{x} = |x^4 - 1|$$

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Be sure to circle the appropriate choice on the answer sheet!

Solution. Squaring we see that $x = x^8 - 2x^4 + 1$ or $x^8 - 2x^4 - x + 1 = 0$. By the Descartes rule of signs, this equation has either 2 or no positive roots. Since $x^8 - 2x^4 - x + 1$ equals 1 for $x = 0$ and -1 for $x = 1$, the equation has at least one, thus 2 roots.

The correct solution is **C.**

9.* A *palindrome* is an integer that reads the same backward as forward; for example 3, 44, 121, 5665, 76467. How many 6 digit palindromes n are there such that n is divisible by 11 and $\frac{n}{11}$ is a palindrome? Write your answer directly on the answer sheet.

Solution. A six digit number divided by 11 can only be a four or a five digit number. The problem is equivalent to finding out how many four or five digit palindromes multiplied by 11 are 6 digit palindromes. Consider first a four digit palindrome $abba$. Multiplication by 11 looks like

$$\begin{array}{cccccc} & & a & b & b & a \\ & & & \times & 1 & 1 \\ \hline & & a & b & b & a \\ & a & b & b & a & \\ \hline x & y & z & z & y & x \end{array}$$

The last digit of the product must be a, so $x = a$. But x is also the leading coefficient of the product, and that can only be 1. So $x = a = 1$. But if $a = 1$, since a is also the leading coefficient of the four digit number, the product cannot have more than five digits. No four digit number fits the bill.

Consider now a five digit palindrome, so the product now looks like

$$\begin{array}{cccccc} & a & b & c & b & a \\ & & & \times & 1 & 1 \\ \hline & a & b & c & b & a \\ a & b & c & b & a & \\ \hline x & y & z & z & y & x \end{array}$$

Looking at the last digit of the product, it must be a; that is $x = a$. We next see that we must have $b+a < 10$; otherwise there is a carry and the first digit of the product would have to be $a+1 \neq x$. Thus $y = a + b$. It is also necessary that $b + c < 10$; otherwise we would not get the second digit of the product to be also y. It is clear that these are all the restrictions; if $abcba$ is a 5 digit number such that $y = a + b < 10$ and $z = b + c < 10$, then $abcba \times 11 = ayzzya$ will be a six digit palindrome. For $1 \le a \le 9$, b can be any value in the range 0 to $9 - a$. For example, if $a = 1$, $b = 8$, we have 2 values of c (0 and 1), for $b = 7$, we have three values of c, and so forth; for $a = 1$ and $b = 0$ we have 10 values of c (all the digits from 0 to 9). Thus for $a = 1$ we have a total of $2 + 3 + \cdots + 10 = 54$ possible solutions. In general we will have $(a + 1) + (a + 2) + \cdots + 10 = (a + 11)(10 - a)/2$ solutions for $a = 1, 2, \ldots, 9$. This works out to

$$54 + 52 + 49 + 45 + 40 + 34 + 27 + 19 + 10 = 330.$$

The correct solution is **330.**

10. Find the positive integer b so that

$$\sum_{k=1}^{2014} \frac{k}{k^4 + k^2 + 1} = \frac{1}{3} + \frac{2}{21} + \frac{3}{91} + \cdots + \frac{2014}{2014^4 + 2014^2 + 1} = \frac{a}{b},$$

where a is an integer and a, b have no positive common divisor other than 1.

(A) 3451 **(B)** 83471 **(C)** 575331 **(D)** 4058211 **(E)** NA

Be sure to circle the appropriate choice on the answer sheet!

Solution. Notice that

$$\frac{k}{k^4 + k^2 + 1} = \frac{1}{2}\left(\frac{1}{k^2 + k + 1} - \frac{1}{k^2 + k + 1}\right)$$

so if $S = \sum_{n=1}^{2014} \frac{k}{k^4+k^2+1}$, then

$$\begin{aligned} 2S &= \sum_{k=1}^{2014} \left(\frac{1}{k^2-k+1} - \frac{1}{k^2+k+1} \right) = \sum_{k=1}^{2014} \left(\frac{1}{k^2-k+1} - \frac{1}{(k+1)^2-(k+1)+1} \right) \\ &= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{13}\right) + \cdots + \left(\frac{1}{2014^2-2014+1} - \frac{1}{2015^2-2015+1} \right) \end{aligned}$$

The second term in every expression in parentheses cancel with the first term of the next expression, so that

$$2S = 1 - \frac{1}{2015^2-2015+1} = 1 - \frac{1}{4058211} = \frac{4058210}{4058211}$$

so that $S = \frac{2029105}{4058211}$
The correct solution is **D.**

11. The sum of all positive integers < 231 that are **not** divisible by 3, or 7, or 11, equals

(A) 10884 **(B)** 11376 **(C)** 13860 **(D)** 14570 **(E)** NA

Be sure to circle the appropriate choice on the answer sheet!

Solution 1, by direct computation. We find the sum of all integers that **are** multiples of at least one of 3,7 or 11, and subtract it from the sum of all integers < 231. If $m < 231$, let S_m denote the sum of all multiples of m that are less than 231. Then

$$\begin{aligned} S_3 &= 3(1+2+\cdots+76) = 3 \cdot \frac{77 \cdot 76}{2} = 38 \times 231, \\ S_7 &= 7(1+2+\cdots+32) = 7 \cdot 33 \cdot 322 = 16 \times 231, \\ S_{11} &= 11(1+2+\cdots+20) = 11 \cdot \frac{21 \cdot 20}{2} = 10 \times 231, \\ S_{21} &= 21(1+2+\cdots+10) = 21 \cdot \frac{11 \cdot 10}{2} = 5 \times 231, \\ S_{33} &= 33(1+2+\cdots+6) = 33 \cdot \frac{7 \cdot 6}{2} = 3 \times 231, \\ S_{77} &= 77(1+2) = 231. \end{aligned}$$

Thus the sum of all integers that **are** multiples of 3, 5, or 7 is

$$S_3 + S_7 + S_{11} - S_{21} - S_{33} - S_{77} = (38 + 16 + 10 - 5 - 3 - 1)231 = 55 \times 231.$$

The sum of all integers < 231 is $\frac{231 \cdot 230}{2} = 115 \times 231$. The number we are looking for is $115 \times 231 - 55 \times 231 = 60 \times 231 = 13860$.

Solution 2 Since $231 = 3 \times 7 \times 11$, ee are trying to find the sum of all integers in the range 1 to 231 relatively prime with 231. A number theorem states that the sum of all positive integers $< n$ relatively prime with n is $n\phi(n)/2$ where ϕ is Euler's totient function; $\phi(n) = \#\{k, 1 \le k \le n, , gcd(k,n) = 1\}$. Since ϕ is multiplicative,

$$\phi(231) = \phi(3)\phi(7)\phi(11) = 2 \times 6 \times 10 = 120,$$

hence the sum of all the integers we are looking for is $231 \times 120/2 = 13860$.

The correct solution is **C.**

12. Let $f(x)$ be a function which contains 2 in its domain and range. Suppose that $f(f(x)) \cdot (1 + f(x)) = -f(x)$ for all x in the domain of $f(x)$. Find $f(2)$.

$$\textbf{(A)}\ -1 \quad \textbf{(B)}\ -\frac{3}{4} \quad \textbf{(C)}\ -\frac{2}{3} \quad \textbf{(D)}\ -\frac{1}{4} \quad \textbf{(E)}\ \text{NA}$$

Be sure to circle the appropriate choice on the answer sheet!

Solution. Since 2 is in the range of the function, there is a such that $2 = f(a)$, Applying the functional equation with $x = a$ gives $f(2)(1+2) = -2$; that is $f(2) = -2/3$.

The correct solution is **C.**

13.* Determine integers a, b, c such that

$$\cos(5x) = a(\cos x)^5 - b(cosx)^3(\sin x)^2 + c(\cos x)(\sin x)^4$$

for all x. Write your answer directly on the answer sheet.

Solution. Let $z = \cos x + i \sin x$. Then

$$\begin{aligned} \cos(5x) &= \text{Re } z^5 = \text{Re } \left(\cos^5 x + 5i \cos^4 x \sin x - 10 \cos^3 x \sin^2 x - 10i \cos^2 x \sin^3 x + 5 \cos x \sin^4 x + i \sin^5 x\right) \\ &= \cos^5 x - 10 \cos^3 x \sin^2 x + 5 \cos x \sin^4 x. \end{aligned}$$

The correct solution is $a = 1, b = 10, c = 5$**.**

14. If the length of each side of a triangle in increased by 20%, then the area of the triangle is increased by

(A) 40% **(B)** 44% **(C)** 48% **(D)** 52% **(E)** 60%

Be sure to circle the appropriate choice on the answer sheet!

Solution. The sides increase by a factor of 1.2, thus the area increases by a factor of $1.2^2 = 1.44$ or by 44%.

The correct solution is **B.**

15. The perimeter of a right triangle is 40 and the sum of the squares of its sides is 578. Find the length of the shortest side.

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Be sure to circle the appropriate choice on the answer sheet!

Solution. Let a, b be the legs of the triangle, $a < b$. We are told that $a + b + \sqrt{a^2 + b^2} = 40$ amd $a^2 + b^2 + (a^2 + b^2) = 589$. From the second equation, $a^2 + b^2 = 289 = 17^2$; using this in the first equation, $a + b = 23$. Then $ab = \frac{1}{2}[(a+b)^2 - (a^2 + b^2)] = \frac{1}{2}(23^2 - 17^2) = 120$, so that a, b are solutions of $x^2 - 23x + 120 = 0$. This equations has the solutions

$$x = \frac{23 \pm \sqrt{23^3 - 480}}{2} = 15, 8.$$

Thus $a = 8$, $b = 15$.

The correct solution is **C.**

16. A primitive pythagorean triple is a triple (a,b,c) of positive integers such that $a^2+b^2=c^2$ **and** such that a,b,c have no divisor in common other than 1 ($\gcd(a,b,c)=1$). How many primitive pythagorean (a,b,c) are there, with $a<b$ such that one of a,b or c equals 660?

(A) 0 **(B)** 3 **(C)** 4 **(D)** 6 **(E)** NA

Be sure to circle the appropriate choice on the answer sheet!

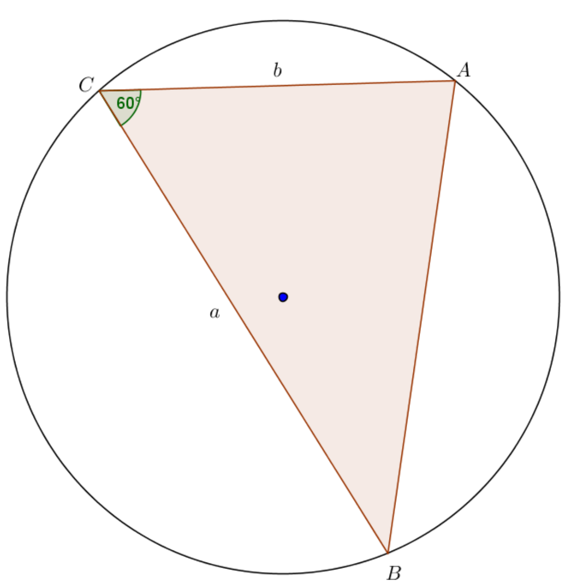
Solution. If (a,b,c) is a primitive pythagorean triple, then one of a,b must be even. We will relabel so b is the even one (so possibly $a>b$). Then (a,b,c) is a primitive p. triple if and only if $a=m^2-n^2, b=2mn, c=m^2+n^2$ for relatively prime integers m,n of opposite parity. Obviously, we need $m>n$. So the question reduces to in how many different ways can we select $1\le n<m$, m,n relatively prime and of different parity such that $660=2mn$ or $330=mn$. Since 330 has precisely 16 divisors that can be paired in the form (m,n) with $m>n$ and, because 4 does not divide 330, m,n of opposite parity. Thus there will be $8=16/2$ such triples. They are:

(a) $n=1, m=330$, producing the triple $(660, 108899, 108901)$.
(b) $n=2, m=165$, producing the triple $(660, 27221, 27229)$.
(c) $n=3, m=110$, producing the triple $(660, 12091, 12109)$.
(d) $n=5, m=66$, producing the triple $(660, 4331, 4381)$.
(e) $n=6, m=55$, producing the triple $(660, 2989, 3061)$.
(f) $n=10, m=33$, producing the triple $(660, 989, 1189)$.
(g) $n=11, m=30$, producing the triple $(660, 779, 1021)$.
(h) $n=15, m=22$, producing the triple $(259, 660, 709)$.

All in all, 8 different primitive pythagorean triples.

The correct solution is **E.**

17.* Triangle ABC is inscribed in a circle of radius $r=4$. If The angle at vertex C is a $60°$ angle and the ratio $x=a/b$ of side a opposite to vertex A to side b opposite vertex B is 1.5, then $a=\dfrac{m\sqrt{p}}{\sqrt{q}}$, where m is an integer and p,q are **odd** prime numbers.



Write out the values of m, p, and q on the answer sheet.

Solution. We recall a simple consequence of the law of sines, the law of tangents: If α, β are the angles at vertices A, B, respectively, then

$$\frac{\tan\frac{\alpha+\beta}{2}}{\tan\frac{\alpha-\beta}{2}} = \frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{x+1}{x-1} = 5.$$

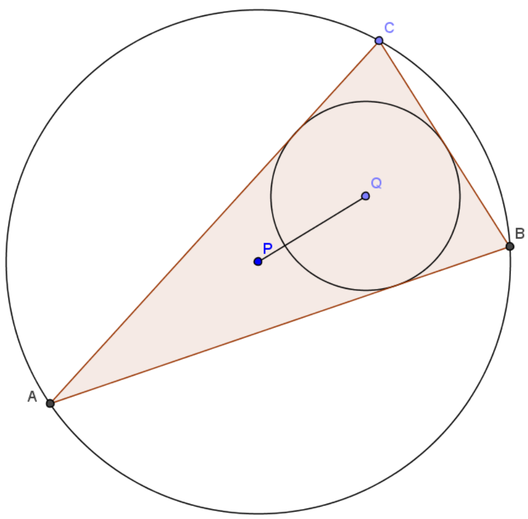
Since $\alpha+\beta+60 = 180$, $\tan((\alpha+\beta)/2) = \tan 60^\circ = \sqrt{3}$ and we get that $\tan((\alpha-\beta)/2) = \sqrt{3}/5$. Then

$$\tan\alpha = \tan\left(\frac{\alpha+\beta}{2} + \frac{\alpha-\beta}{2}\right) = \frac{\tan\frac{\alpha+\beta}{2} + \tan\frac{\alpha-\beta}{2}}{1 - \tan\frac{\alpha+\beta}{2}\cdot\tan\frac{\alpha-\beta}{2}} = 3\sqrt{3}.$$

Then $\sin\alpha = 3\sqrt{3}/(2\sqrt{7})$; by the law of sines $a = 2r\sin\alpha = 12\sqrt{3}/\sqrt{7}$.

The correct solution is $m = 12, p = 3, q = 7$**.**

18. A circle of center Q is inscribed in a triangle ABC and a circle of center P circumscribes the same triangle.



If the distance $|PQ|$ between the centers of the circles is 4 and the radius of the inscribed circle (of center Q) is 3, then the radius of the circumscribed circle is:

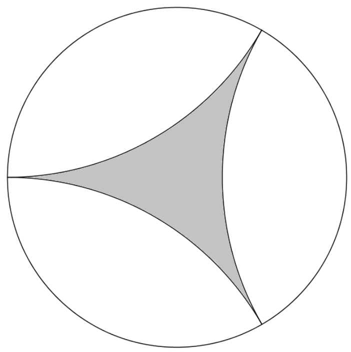
(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** NA

Be sure to circle the appropriate choice on the answer sheet!

Solution. By a theorem due to Euler, given circles C, C' of radii R, r, respectively, with C' included in C, there exists a triangle circumscribed to C' and inscribed in C if and onlt if the distance d between the centers of C, C' satisfies $d^2 = R^2 - 2rR$. In our case this works out to $R^2 - 6R - 16 = 0$. This equation has two roots, 8 and -2. Since the radius must be positive, the answer is 8.

The correct solution is **D.**

19. The sides of the shaded "triangular" figure are equal length arcs intersecting the circle circumscribing the figure in right angles. (An arc intersects a circle in a right angle if at the point of intersection the tangent to the circle is perpendicular to the tangent to the arc)

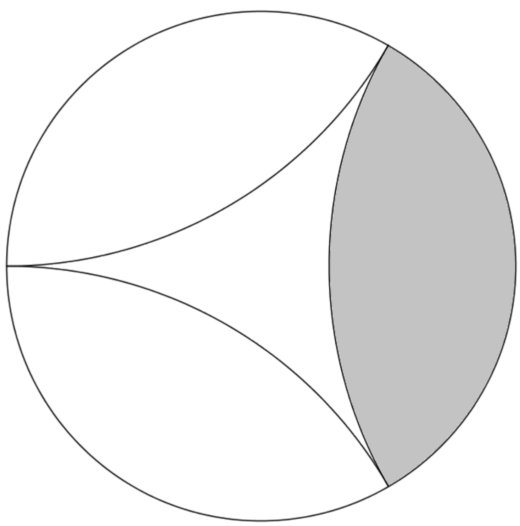


Determine the area of the shaded figure.

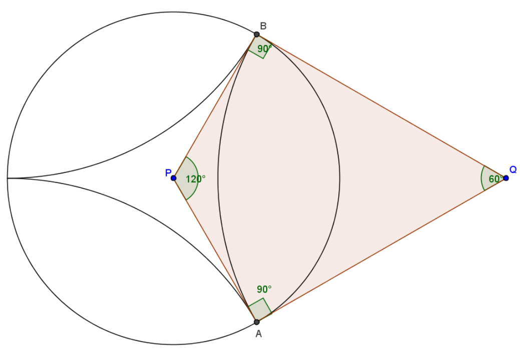
(A) $3\left(\sqrt{3}-\frac{\pi}{2}\right)r^2$ **(B)** $4\left(\sqrt{2}-\frac{\pi}{3}\right)r^2$ **(C)** $\frac{1}{3}\left(\sqrt{3}+\frac{\pi}{2}\right)r^2$ **(D)** $\frac{1}{4}\left(\sqrt{2}+\frac{\pi}{3}\right)r^2$ **(E)** NA

Be sure to circle the appropriate choice on the answer sheet!

Solution. First we find the area of one of the lens shaped figures; shaded in the picture below



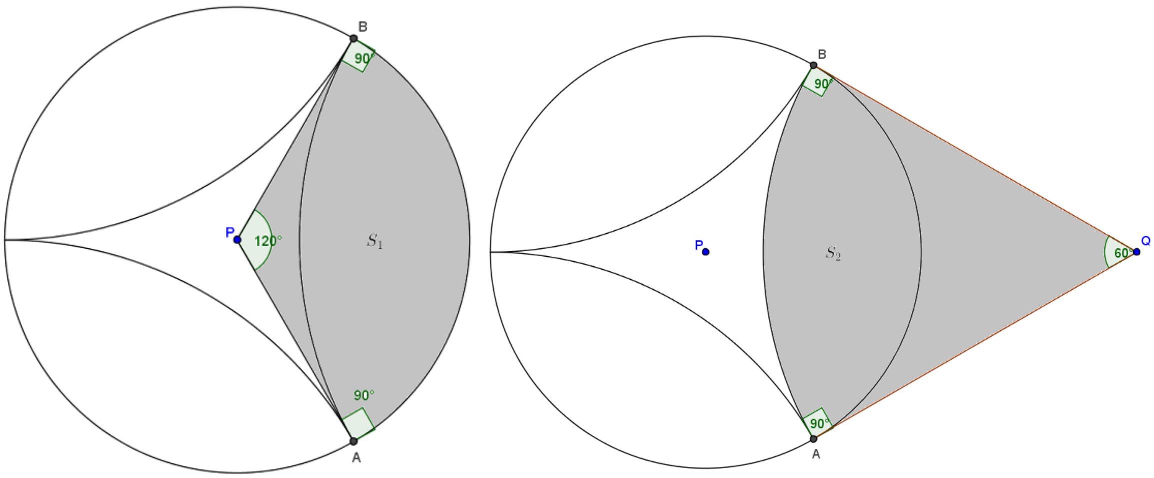
Because the arc intersects the circle of radius r at right angles, the tangents to that circle at the points of intersection must meet at the center of the circle of which the arc is part. The next picture shows the tangents. We have also drawn the radii from the centers P, Q of the two circles to the points of intersection (points A, B).



We have $|PA| = |PB| = r$, and one easily sees that $|QA| = |QB| = \sqrt{3}r$. It follows that the area of the kite shaped figure pictured above is

$$A(K) = \sqrt{3}\,r^2.$$

Now we consider the two triangular sectors S_1, S_2 pictured below



They work out to

$$A(S_1) = \frac{1}{2}r^2\frac{2\pi}{3} = \frac{\pi}{3}r^2, \qquad A(S_2) = \frac{1}{2}(\sqrt{3}r)^2\frac{\pi}{3} = \frac{\pi}{2}r^2.$$

The lens shaped figure is $S_1 \cap S_2$ and since $K = S_1 \cup S_2$, the area of the lens shaped figure works out to

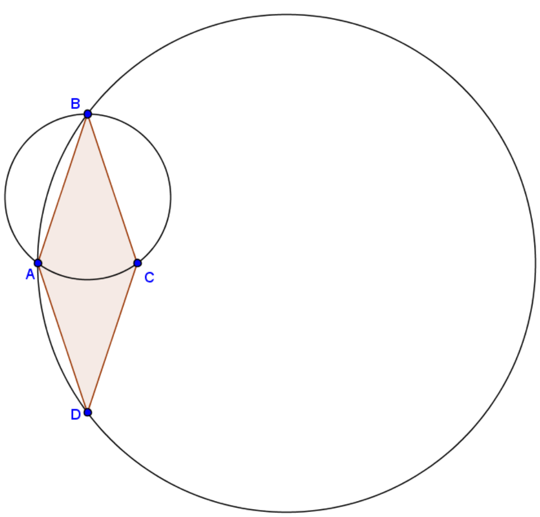
$$A(S_1 \cap S_2) = A(S_1) + A(S_2) - A(S_1 \cup S_2) = \frac{1}{2}r^2\frac{2\pi}{3} + \frac{\pi}{2}r^2 - \sqrt{3}\,r^2 = \left(\frac{5\pi}{6} - \sqrt{3}\right)r^2.$$

The area of the triangle "triangle" will equal the area of the circle minus 3 times the area of one of these "lenses;" that is

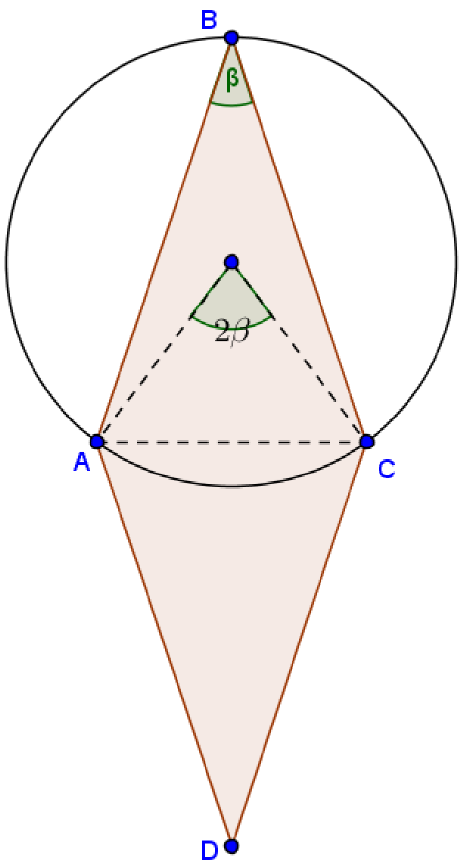
$$A = \pi r^2 - 3\left(\frac{5\pi}{6} - \sqrt{3}\right)r^2 = 3\left(\sqrt{3} - \frac{\pi}{2}\right)^2.$$

The correct solution is **A.**

20.* A *rhombus* is a quadrilateral with all 4 sides equal. Consider a rhombus of vertices A, B, C, D. The circle through A, B, C has radius 5, the circle through ABD has radius 15. Find the area of the rhombus (it is an integer); write your answer directly on the answer sheet.



Solution. Let ℓ be the length of the side of the rhombus, α the angle at A, β the angle at B. Notice that $\alpha + \beta = \pi$. We consider the half of the rhombus in the smaller circle. We find a general formula, so assume the radius of the smaller circle is r, of the larger circle R.



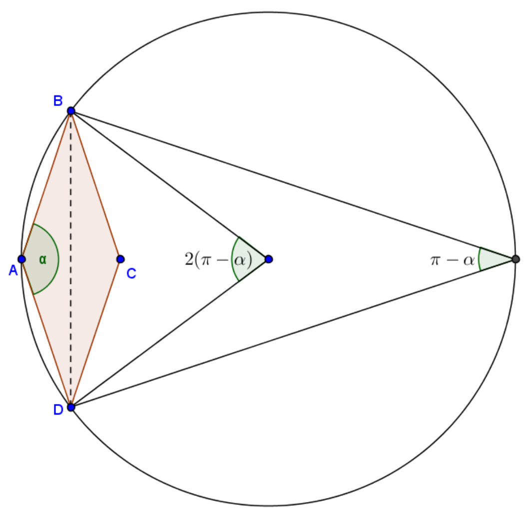
A bit of trigonometry shows that

$$\sin\beta = \frac{|AC|}{2r}, \qquad \sin\frac{\beta}{2} = \frac{|AC|}{2\ell}$$

so that

$$\cos\frac{\beta}{2} = \frac{\sin\beta}{2\sin(\beta/2)} = \frac{\ell}{2r}.$$

Similar considerations hold for the half ABD of the rhombus in the larger circle, except that the angle α is obtuse.



We get

$$\sin(\pi - \alpha) = \frac{|BD|}{2R}, \sin\frac{\alpha}{2} = \frac{|BD|}{2\ell};$$

since $\sin(\pi - \alpha) = \sin\alpha$ we get the formula analogous to the one for the smaller circle, namely

$$\cos\frac{\alpha}{2} = \frac{\ell}{2R}.$$

Since $\alpha = \pi - \beta$ we have $\cos\alpha 2 = \sin\beta 2$ so that

$$\frac{\ell}{2R} = \sqrt{1 - \frac{\ell^2}{4r^2}},$$

squaring and solving for ℓ we get

$$\ell^2 = \frac{4R^2r^2}{R^2 + r^2}.$$

Recall now the formula for the area of a triangle of sides a, b, c inscribed in a circle of radius r. It is $abc/(4r)$. We thus have, denoting by A the area of the rhombus:

$$\begin{aligned} A &= 2 \times \text{area of} \triangle ABC = 2 \cdot \frac{|AC|\ell^2}{4r} = \frac{|AC|\ell^2}{2r} = \frac{2R^2r|AC|}{R^2 + r^2}, \\ A &= 2 \times \text{area of} \triangle ABD = \frac{2Rr^2|BD|}{R^2 + r^2}. \end{aligned}$$

Multiplying the two expressions together,

$$A^2 = \frac{4R^3r^3|AC| \cdot |BD|}{(R^2 + r^2)^2}.$$

Noticing that the area of $\triangle ABC$ equals $|AC| \cdot |BC|/2$ so that $|AC| \cdot |BC| = 2A$ we see that

$$A = \frac{8R^3r^3}{(R^2 + r^2)^2}.$$

Specializing to $r = 5, R = 15$, we get $A = 54$.

The correct solution is **54.**