MATH DAY 2013 at FAU

Competition A–Individual
SOLUTIONS

NOTE: Two of the problems had problems. A typo occurred in the choice of possible answers for Problem # 6 and Problem # 16 had two possible correct answers. The solutions to these problems explain how these issues were resolved.

1. Of 4 tests, each out of 100 points, my average is 87. What is the lowest possible score I can have on any of these tests?

   (A) 0 **(B)** 48 **(C)** 80 **(D)** 86 **(E)** NA

   Solution. The answer is B.

   To have an average of 87, your test scores must add up to $4 \times 87 = 348$. Clearly, 48 is the lowest score you can have in one test and still hope to have your score add up to 348 by getting 100 on all remaining tests.

2. A lattice point in the plane is a point both of whose coordinates are integers. How many lattice points (including the endpoints) are there on the line segment joining the points $(2, 0)$ and $(16, 203)$?

   (A) 15 **(B)** 14 **(C)** 9 **(D)** 8 **(E)** NA

   Solution. The answer is D.

   The line has slope 203/14 so its equation is $y = \dfrac{203}{14}(x-2)$. If (m,n) is a lattice point on the line segment joining the two points in question, we will have $2 \le m \le 16$ and $n = \dfrac{203}{14}(m-2)$, equivalently $203m - 14n = 406$. There are stabdard ways of finding all integer solutions of equations such as this one, but they are hardly needed here. If one realizes that all coefficients are divisible by 7, one can rewrite the equation (after dividing by 7 and rearranging) in the form $2n = 29m - 58$. It should be clear that m cannot be odd; then $29m$ is odd and can't equal $2n + 58$. On the other hand, if m is even, then $29m - 58$ will be even, and expressible in the form $2n$. Thus there will be a solution for every even m, $2 \le m \le 16$; a total of 8 solutions.

3. A certain function f satisfies $f(x) + 2f(6-x) = x$ for all real numbers x. Find $f(1)$.

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution. The answer is D, 3. The equation, with $x = 1$ and $x = 5$ gives

   $$\begin{aligned} f(1) + 2f(5) &= 1 \\ f(5) + 2f(1) &= 5 \end{aligned}$$

   Solving, we get $f(1) = 3$.

4. Given $y > 0$, $x > y$, and $z \neq 0$. Which of the following inequalities is **not** always correct?

   (A) $x - z > y - z$ **(B)** $xz > yz$ **(C)** $\dfrac{x}{z^2} > \dfrac{y}{z^2}$ **(D)** $xz^2 > yz^2$ **(E)** NA

   Solution. B can be false.

   Inequality B fails if $z < 0$. All others hold for all values of x, y, z satisfying $0 < y < x$, $z \neq 0$.

5. Barton and Gabrielle are brother and sister. Barton has twice as many sisters as brothers. Gabrielle has twice as many brothers as sisters. The number of girls in the family is

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution. The answer is A, 2.

Let g be the number of girls, b of boys, in the family. That Barton has twice as many sisters as brothers can be expressed by $g = 2(b-1)$. Similarly, that Gabrielle has twice as many brothers as sisters can be symbolized by $b = 2(g-1)$. Substituting b in the first equation by its value in the second equation, we get $g = 2(2(g-1)-1) = 4g-6$, thus $g = 2$.

6. What condition on the real number a is necessary and sufficient so that the equation

$$|x-1| + |x-2| = a$$

has exactly two real roots?

(A) $a > 1$ **(B)** $a = 1$ **(C)** $1 \leq a$ **(D)** $1 < a$ **(E)** NA

Solution. The intended option A was $a > 0$, so that the correct answer would be D, $a > 1$. After some discussion it was decided that because the correct solution was among the options (in fact, twice), one could not give credit for choosing option E. Moreover, students selecting option A (or D) were probably choosing it because they either may not have seen that A and D were the same, or decided they had to simply make a choice. The bottom line (to use that overused expression) is that anybody circling A or D had the right answer, so either answer was graded as right.

The best approach might be to solve the equation. Assuming $x \geq 2$, the equation is $x-1+x-2 = a$ or $2x-3 = a$, thus $x = (a+3)/2$. But this makes sense only if $(a+3)/2 \geq 2$ or $a \geq 1$. If $1 \geq x < 2$, then the equation becomes $x-1+2-x = a$ or $1 = a$. There are thus no solutions if $a \neq 1$, and infinity of solutions if $a = 1$. If $x < 1$, the equation becomes $1-x+2-x = a$ or $3-2x = a$, thus $x = (3-a)/2$. To make sense we need to have $(3-a)/2 < 1$, which is equivalent to $a > 1$. If $a > 1$ we have precisely two solutions, namely $(3 \pm a)/2$; if $a = 1$ we have an infinity of solutions, every value of x in the interval $1 \leq x \leq 2$; if $a < 1$ there are no solutions.

7. Five students are to sit from left to right for a photo. One of them is the tallest. How many seating arrangements are possible if the tallest student cannot sit at the leftmost or the rightmost?

(A) 36 **(B)** 72 **(C)** 96 **(D)** 120 **(E)** NA

Solution. The answer is B, 72.

There will be three positions for the tallest student. For each one of these, the remaining four students can sit in any position they want, so in a total of $4! = 24$ different positions. The total number of positions is $3 \times 24 = 72$

8. Consider the following sequence

$$-, +, +, -, +, +, \cdots$$

where every minus sign is followed by two plus signs. How many minus signs are there in the first 2014 terms?

(A) 670 **(B)** 671 **(C)** 672 **(D)** 673 **(E)** NA

Solution. The answer is C, 672

There is one minus sign every 3 terms; since 2013 is a multiple of 3, there are $2013/3 = 671$ minus signs so far. The 2014th sign will again be a minus sign, so there will be $671 + 1 = 672$ minus signs in the first 2014 terms.

9.* Find the smallest positive integer that has a remainder of 3 when divided by 4, of 4 when divided by 5, and of 1 when divided by 7. Write the answer directly onto the answer sheet.

Solution. The answer is 99.

Let x be the integer we look for. Having a remainder of 3 when divided by 4 means x is of the form $x = 3 + 4k$ for some integer k. Now $3 + 4k$ has a remainder of 4 when divided by 5, so $3 + 4k = 4 + 5j$ for some integer j, thus $4k = 1 + 5j$. If j were even, $4k = 1 + 5j$ would be odd,which is impossible. So j must be odd, say $j = 2j' + 1$. Then $4k = 6 + 10j'$, $2k = 3 + 5j'$, and again j' has to be odd, say $j' = 2m + 1$. Then $2k = 8 + 10m$, thus $k = 4 + 5m$, and $x = 19 + 20m$. To satisfy that the remainder of x divided by 7 is 1, we now need $19 + 20m = 1 + 7n$ for some integer n.

Thus $20m = -18 + 7n$. The fact that we have a negative number (-18) can be remedied by replacing, as we may, n by $n+3$ and (relabeling) writing $20m = 3 + 7n$. Now we can try to find the smallest n making this possible, and that's $n = 11$; $3 + 77 \cdot 11 = 80 = 4 \cdot 20$. With $m = 4$ we have $x = 19 + 80 = 99$.

This exercise is much easier to do if you know congruences, and heard about the Chinese remainder theorem.

10. With i denoting the imaginary unit $\sqrt{-1}$, let a, b be real numbers such that

$$(a + bi)^2 = 16 + 30i.$$

If $a > 0$, then a equals

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution. The answer is D, 5.

Squaring $a+bi$ and equating the real and imaginary parts of the result to those of $16+30i$, we get $a^2-b^2 = 16, 2ab = 30$. Thus $b = 15/a$ and

$$a^2 - \frac{225}{a^2} = 16, \quad \text{thus} \quad a^4 - 16a^2 - 225 = 0$$

This is a quadratic equation in a^2 with solutions

$$a^2 = \frac{16 \pm \sqrt{256 + 900}}{2} = 8 \pm \frac{1}{2}\sqrt{1156} = 8 \pm \sqrt{\frac{1156}{2}} = 8 \pm \sqrt{289} = 8 \pm 17.$$

Because a is real, we must have $a^2 \geq 0$, so we conclude that $a^2 = 8 + 17 = 25$ thus, since $a > 0$, $a = 5$.

11. With i denoting the imaginary unit $\sqrt{-1}$, assume a, b are real numbers such that

$$(a + bi)^5 = 41 + 38i.$$

Find $a^2 + b^2$.

(A) 5 **(B)** 79 **(C)** $\sqrt[5]{79}$ **(D)** $\sqrt[5]{158}$ **(E)** NA

Solution. The answer is A, 5.

If $z = c + di$ is a complex number, c, d real, then $|z|$ denotes the absolute value of z, $|z| = \sqrt{c^2 + d^2}$. Since the absolute value is multiplicative,

$$|41 + 38i|^2 = |(a + bi)^5|^2 = |(a + bi)^2|^5 = (a^2 + b^2)^5;$$

now

$$|41 + 38i|^2 = 41^2 + 38^2 = 1681 + 1444 = 3125 = 5^5,$$

so that $a^2 + b^2 = 5$.

Note: How does one recognize that $3125 = 5^5$ without a calculator? Having to find roots of a number gets simplified if one can factor the number, so a logical first step is to try to factor it. Clearly 3125 is divisible by 5, the result is 625, which is again divisible by 5; etc.

12. The roots of the equation $2x^3 - 9x^2 - 11x + c = 0$ are in arithmetic progression. Determine c

(A) 30 **(B)** 25 **(C)** 20 **(D)** 15 **(E)** 10

Solution. The answer is A, 30. Let $x_1 = a$, $x_2 = a + r$, $x_3 = a + 2r$ be the roots. By Viète's relations

$$\begin{aligned} \frac{9}{2} &= x_1 + x_2 + x_3 = 3a + 3r, \\ -\frac{11}{2} &= x_1x_2 + x_1x_3 + x_2x_3 = 3a^2 + 6ar + 2r^2. \end{aligned}$$

From the first equation, $r = (9 - 6a)/6$; substituting into the second equation, and simplifying, $a^2 - 3a - 20 = 0$. This last equation has the solutions $-2, 5$. Selecting $a = -2$ we get $r = 7/2$ and the roots work out to $-2, 3/2, and 5$. Selecting $a = 5$ we get $r = -7/2$, and the roots come out the same, but in decreasing order. By Viète's relation,

$$\frac{c}{2} = -x_1x_2x_3 = -(-2)(\frac{3}{2})(5) = 15, \quad \text{thus} \quad c = 30.$$

13.* How many numbers of the form $1^n + 2^n + 3^n + 4^n$ are divisible by 5, if n is an integer, $100 \leq n \leq 200$?

Solution. The answer is 75.

Recall Fermat's Little Theorem, $a^{p-1} \equiv 1 \pmod{p}$, if p is prime and p does not divide a. It follows that $a^4 \equiv 1 \pmod{5}$ for $a = 1, 2, 3, 4$. Given any integer n, we can divide it by 4 and write it in the form $n = 4q + r$, where $r = 0, 1, 2,$ or 3. Thus

$$1^n + 2^n + 3^n + 4^n = (1^4)^q 1^r + (2^4)^q 2^r + (3^4)^q 3^r + (4^4)^4 4^r \equiv 1^r + 2^r + 3^r + 4^r \pmod{5}.$$

By direct computation

$$1^r + 2^r + 3^r + 4^r = \begin{cases} 4 \not\equiv 0 \pmod{5} & \text{if } r = 0 \\ 10 \equiv 0 \pmod{5} & \text{if } r = 1 \\ 30 \equiv 0 \pmod{5} & \text{if } r = 2 \\ 100 \equiv 0 \pmod{5} & \text{if } r = 3 \end{cases}$$

It follows that $1^n + 2^n + 3^n + 4^n$ is divisible by 5 if and only if n does not have remainder 0 when divided by 4; that is if and only if n is not divisible by 4. In the range 100-200 there are 101 numbers of which 26 are divisible by 4. That leaves 75 numbers that are not divisible by 4.

14. What are the last three digits of 3^{2013}?

(A) 123 **(B)** 323 **(C)** 523 **(D)** 723 **(E)** 923

Solution. The answer is B.

The question may be rephrased as asking to what three digit number 3^{2013} is congruent mod 1000. It may help to use Euler's Theorem,

$$a^{\phi(n)} \equiv 1 \pmod{n}$$

if a and n are relatively prime (have no common divisors other than 1), where ϕ is Euler's totient function; $\phi(n) =$# of numbers m, $1 \leq m < n$, such that m and n are relatively prime. It also helps to know that there is a formula for ϕ; if p is prime, then $\phi(p^k) = p^k - p^{k-1}$ and if a, b are relatively prime, then $\phi(ab) = \phi(a)\phi(b)$. Thus

$$\phi(1000) = \phi(2^3 \times 5^3) = (8 - 4)(125 - 25) = 400.$$

It follows that $3^{400} \equiv 1 \pmod{1000}$, hence

$$3^{2013} = (3^{400})^5 \cdot 3^{13} \equiv 3^{13} \qquad (\text{mod} 1000).$$

It takes only a few computations to see that $3^{13} \equiv 323 \pmod{1000}$.

15. Which of the following numbers is closest to the number of digits of 2^{125}?

(A) 25 **(B)** 30 **(C)** 35 **(D)** 40 **(E)** 45

Solution. D, 40.

Because $2^{10} = 1024$ is close to 10^3, a good way of estimating a power of 2 is to replace 2^n by $10^{n/3}$. That is

$$2^{125} = \left(2^{10}\right)^{12.5} \sim 10^{37.5}$$

We can expect the number of digits to be at least 38 (since 10^{37} already has 38 digits), but not larger than 40.

The exact number is, in fact, 38. One can see this, without need of calculators, by noticing for example that

$$2^{10} = 1024 = 1000 \times 1.024 = 1000 \times \left(1 + \frac{24}{1000}\right) < 1000 \times \left(1 + \frac{25}{1000}\right) = 10^3 \times \left(1 + \frac{1}{40}\right).$$

Thus

$$2^{125} = \left(2^{10}\right)^{12.5} < 10^{37.5} \left(1 + \frac{1}{40}\right)^{37.5} = 10^{37}\sqrt{10} \left(1 + \frac{1}{40}\right)^{37.5}.$$

Now $\left(1 + \frac{1}{40}\right)^{37.5} < \left(1 + \frac{1}{40}\right)^{40} < e$. Since $e\sqrt{10} < 3\sqrt{10} < 10$, we see that

$$2^{125} < 10^{38},$$

thus cannot have more than 38 digits.

16. Abe is a compulsive hoarder of Lincoln pennies. He has more than 10,000 pennies filling three boxes. One fifth (1/5) of his pennies are stashed in one of these boxes. A second box is divided into a number of compartments, each compartment contains one nineteenth (1/19) of his collection. Finally, the third box contains 2013 pennies. How many pennies does he have in all?

Solution. There are two possible right answers. The problem should have stated to find the smallest possible number of pennies. The answers are: $17,385$ pennies or $191,235$ pennies. Students getting either of these two answers were graded as having answered correctly the question.

Let x be the total number of pennies. Let a be the number of compartments in the second box. Then

$$\begin{aligned} \#\text{ of pennies in box 1:} &= \frac{1}{5}x, \\ \#\text{ of pennies in box 2:} &= \frac{1}{19}x, \\ \#\text{ of pennies in box 3:} &= 2013, \end{aligned}$$

Adding up,

$$\frac{1}{5}x + \frac{a}{19}x + 2013 = x.$$

If we solve this equation for x we get

$$x = \frac{2013 \times 95}{76 - 5a}.$$

For this equation to make sense, in the first place we must have $5a < 76$, limiting a to be one of $1, 2, \ldots, 15$. But $76 - 5a$ has to divide $2013 \times 95 = 3 \times 5 \times 11 \times 19 \times 61$ so $76 - 5a$ must be one of 1, 3, 5, 11, 15, 19, 33, 55, 57, 61. But 5 cannot possibly be a factor of $76 - 5a$, which eliminates not only 5 but also 15 and 55. Since a will have to be odd, and the last digit of $76 - 5a$ will then be 1, we can also eliminate 3, 19, 33, and 57. This leaves 1, 11, and 61. If $a = 15$, we get $76 - 5a = 1$. If $a = 13$ we get $76 - 5a = 11$. If $a = 3$ we get $76 - 5a = 61$. Now,

$$\begin{aligned} \text{With } a = 3,\ x &= \frac{2013 \times 95}{61} = 3135, \\ \text{With } a = 13,\ x &= \frac{2013 \times 95}{11} = 17,385, \\ \text{With } a = 15,\ x &= \frac{2013 \times 95}{1} = 191,235. \end{aligned}$$

17. Find the remainder of dividing $x^{101} + 3x^{74} + 4x^{26} + 7x^2$ by $x^5 + x^2$.

(A) $x^4 - 3x^2$ **(B)** $17x^3 + x^2$ **(C)** $9x^3 - 5x^2$ **(D)** $13x^2$ **(E)** NA

Solution. D, $13x^2$.

Let $P(x) = x^{101} + 3x^{74} + 4x^{26} + 7x^2 = x^2(x^{99} + 3x^{72} + 4x^{24} + 7)$. The trick is to notice that $P(x) = x^2Q(x^3)$, where $Q(x) = x^{33} + 3x^{24} + 4x^8 + 7$. By the division algorithm, the remainder of dividing $Q(x)$ by $x + 1$ is $Q(-1) = -1 + 3 + 4 + 7 = 13$; that is $Q(x) = (x + 1)G(x) + 13$ for some polynomial $G(x)$. Thus

$$P(x) = x^2Q(x^2) = x^2[(x^3 + 1)G(x^3) + 13] = (x^5 = x^2)G(x^3) + 13x^2.$$

Since the degree of $13x^2$ is less than 5, which is the degree of $x^5 + x^2$, it is the remainder.

18.⋆ Find the sum of all distinct three digit numbers containing only the digits 1, 2, 3, and 4, each no more than once.

Solution. The answer is 6660. There are 24 such numbers, one could just write them all out and add them. But a faster (and safer) way is by noticing that the there will be exactly 6 numbers starting with the digit 1, 6 with the digit 2, 6 with 3, and 6 with 4. The sum of the first digits will be $6 + 12 + 18 + 24 = 60$. The same pattern repeats for the second and third digits, so that the sum is $60 \times (100 + 10 + 1) = 6660$.

19. How many real roots does the equation $\log_{10} x = \sin x$ have? (In $\sin x$, x is measured in radians.)

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** NA

Solution. The answer is C, 3.

$\log_{10} x$ is strictly increasing, negative for $x < 1$. It intersects the graph of $\sin x$ once for $1 < x < \pi$, then a second and third time for $2\pi < x < 3\pi$. After 3π, $\sin x$ is negative and stays negative past $x = 10$, after which $\log_1 0x > 1$ and can't equal $\sin x$ anymore.

20. A triangle has sides of lengths $a, 3a/2$, and $2a$, where a is a positive real number. If the area is A and the square of the area is $A^2 = 15$, then a equals

(A) 4/3 **(B)** 3/4 **(C)** 2/3 **(D)** 3/2 **(E)** NA

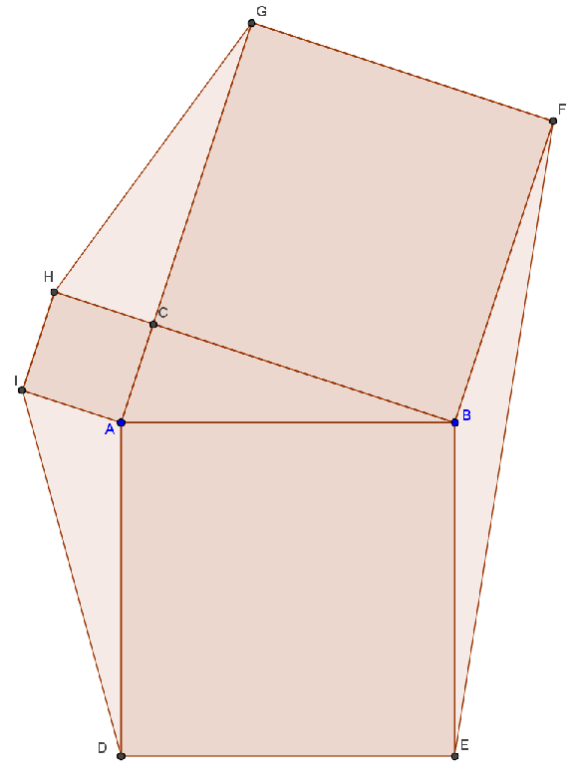
Solution. The answer is E, all of A, B,C,D are wrong.

By Heron's formula, for a triangle of sides a, b, c, $A = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = (a + b + c)/2$ is the semi-perimeter. In our case this works out to

$$A^2 = \frac{135a^4}{256} = 15,$$

thus $a^4 = 256/9 = 4/\sqrt{3}$.

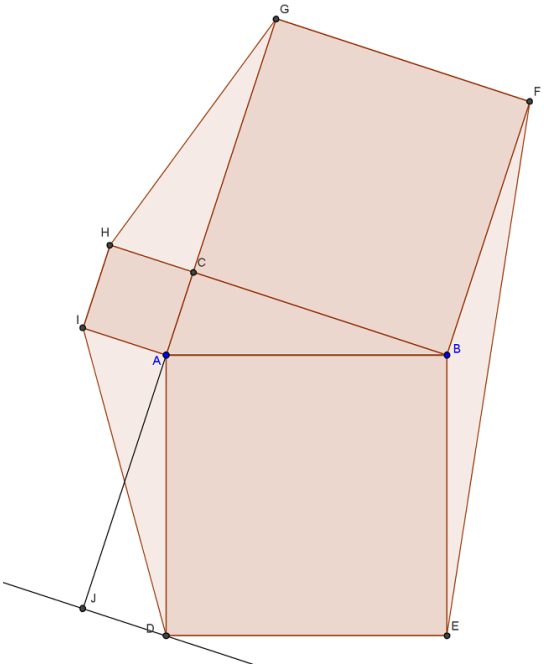
21.⋆ An irregular hexagon $DEFGHI$ is drawn as follows: We start with a right triangle ABC, draw the squares on the legs and hypotenuse, and then join in sequence the free vertices of the squares.



If the hypotenuse has **length** 13 and the triangle ABC has **area** 25, find the area of the hexagon $DEFGHI$. Write the answer directly onto the answer sheet.

Solution. The correct answer is 438.

Consider the picture

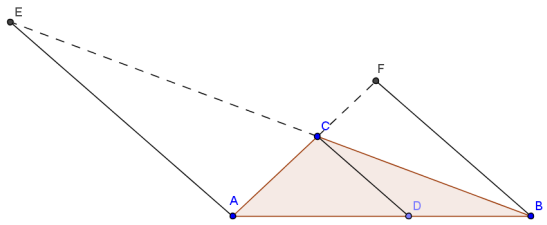


where we drew through D a line parallel to IA (and HB), and continued the segment CA until it intersects the line through D at J. It is easy to see that the triangle AJD is a right triangle congruent to the triangle ABC. Thus its area is also 25. But this triangle shares with the triangle ADI a base and an altitude, thus the triangle ADI has the

same area of 25 as AJD. Similarly one sees that the triangle FBE has area 25. The total area of the hexagon is thus 4 times the area of ABC plus the sum of the areas of the squares, thus

$$|AB|^2 + |BC|^2 + |CA|^2 + 4 \times |ABC| = 13^2 + 13^2 + 4 \times 25 = 438.$$

22. A line segment is drawn from a point D on side AB of triangle ABC to the opposite vertex C. Lines parallel to DC are drawn through A and B; the extension of side BC intersects the first parallel at E, the extension of AC intersects the second parallel at F.



If $|AE| = 6$ and $|BF| = 4$, what is $|DC|$?

(A) 2 **(B)** 2.2 **(C)** 2.4 **(D)** 2.6 **(E)** 2.8

Solution. The answer is C, 2.4

Triangles ABE and DBC are similar, thus $\dfrac{|DC|}{|AE|} = \dfrac{|DB|}{|AB|}$. Similarly, triangles BAF and DAC are similar so that $\dfrac{|DC|}{BF} = |AD||AB|$. Adding

$$\frac{|DC|}{|AE|} + \frac{|DC|}{BF} = \frac{|DB|}{|AB|} + |AD||AB| = \frac{|AD| + |DB|}{|AB|} = \frac{|AB|}{|AB|} = 1;$$

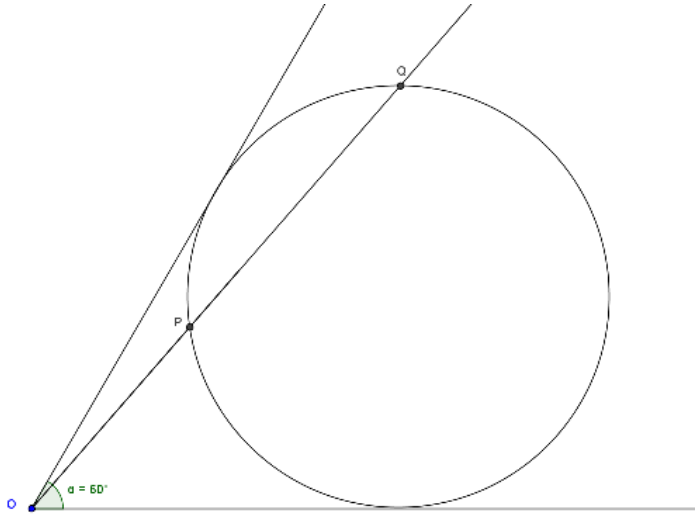
that is,

$$1 = \frac{1}{6}|DC| + \frac{1}{4}|DC| = \frac{10}{24}|DC|,$$

thus $|DC| = 24/10 = 2.4$

23. A circle of radius r is between and tangent to two lines intersecting at O at an angle of $60°$. A third line drawn from O intersects the circle at points P and Q. Determine $|OP|$, the length of the segment OP, if $|PQ| = r/2$.

(A) r **(B)** $3r/2$ **(C)** $2r$ **(D)** $5r/2$ **(E)** NA



Solution. The answer is B, $|OP| = 3r/2$.

Let S be the point where the circle is tangent to the horizontal line; then $|OS| = r\cot 30° = \sqrt{3}\,r$. By the constancy of the power of points with respect to the circle, we have $|OP| \cdot |OQ| = |OS|^2$; that is $|OP|(|OP| + r/2) = 3r^2$. This yields the following quadratic equation for $|OP|$:

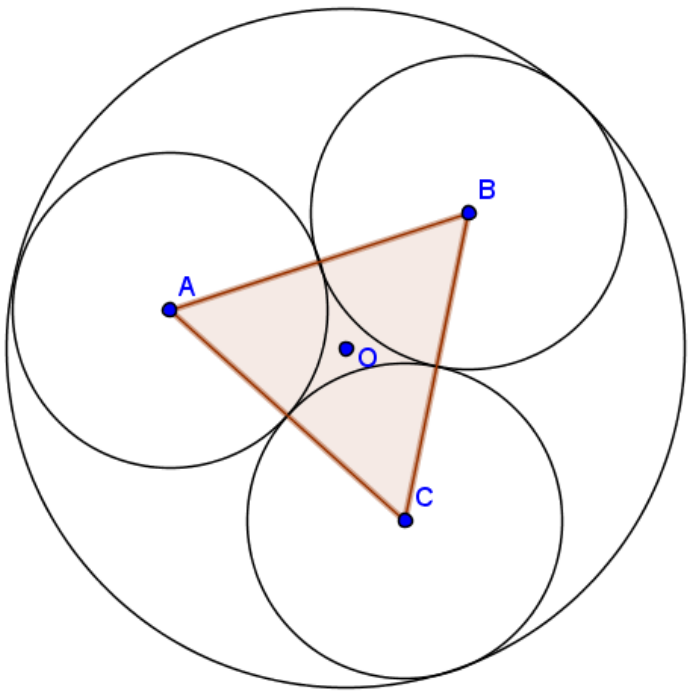
$$|OP|^2 + \frac{r}{2}\,|OP| - 3r^2 = 0.$$

The solutions are

$$|OP| = \frac{-\frac{r}{2} \pm \frac{7r}{2}}{2}.$$

Ignoring the negative solution, we get $|OP| = 3r/2$.

24. Three circles of equal radius r are drawn inside a large circle of radius R so that they all are tangent to each other, and to the large circle.



The relation between r and R can be expressed in the form

$$r = \frac{\sqrt{a}}{b + \sqrt{a}} R$$

where a, b are prime numbers. What is $a^2 + b$?

(A) 7 **(B)** 11 **(C)** 14 **(D)** 28 **(E)** NA

Solution. The answer is B, 11.

The easiest way of solving this problem is (probably) using a bit of symmetry. For reasons of symmetry, the triangle ABC is equilateral and the center O of the large circle will be the circumcenter of the triangle. The sides of the triangle have length $2r$, so that the radius ρ of the circumcenter satisfies

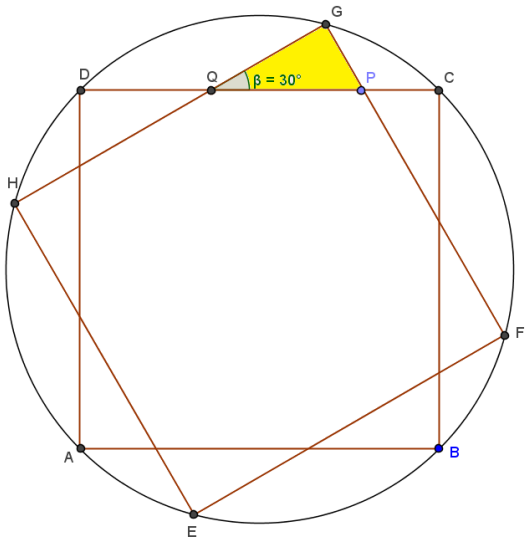
$$2\rho = \frac{2r}{\sin 60^\circ} = \frac{4r}{\sqrt{3}};$$

thus $r = \rho\sqrt{3}/2$. It is also clear from the picture that $\rho = R - r$ so that we have $r = \dfrac{(R-r)\sqrt{3}}{2}$. Solving for R we get

$$r = \frac{\sqrt{3}}{2 + \sqrt{3}} R,$$

so $a = 3, b = 2$ and $a^2 + b = 11$.

25. Squares $ABCD$ and $EFGH$ are inscribed in a circle in such a way that one side of square $EFGH$ intersects a side of $ABCD$ forming an angle of 30°, as shown.



If the inscribed squares have sides of length ℓ (so, for example, $|AB| = \ell$), determine the area of the triangle QPG.

(A) $\dfrac{3}{64}\ell^2$ **(B)** $\dfrac{3\sqrt{2}-3}{12}\ell^2$ **(C)** $\dfrac{1}{12}\ell^2$ **(D)** $\dfrac{2\sqrt{2}-2}{16}\ell^2$ **(E)** $\dfrac{2\sqrt{3}-3}{12}\ell^2$

Solution. The answer is E, $\dfrac{2\sqrt{3}-3}{12}\ell^2$.

It is clear that the eight corner triangles are similar right triangles. It is also clear that all their altitudes are the same, which (being similar) forces them to be congruent. If $a > b$ denote the legs of these corner triangles and c the hypotenuse, we have the following relations among them:

$$\begin{aligned} a+b+c &= \ell \qquad \text{(All corner triangles are congruent)} \\ 2b &= c \qquad (b/c = \cos\angle PQG = \cos 30^\circ = 1/2) \\ a^2+b^2 &= c^2 \qquad \text{(Pythagoras)} \end{aligned}$$

From these relations we can solve for a, b geting

$$\begin{aligned} a &= \frac{\sqrt{3}}{2}\ell - \frac{1}{2}\ell, \\ b &= \frac{1}{2}\ell - \frac{\sqrt{3}}{6}\ell. \end{aligned}$$

Thus the area is

$$A = \frac{1}{2}ab = \frac{1}{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right)\ell^2 = \frac{2\sqrt{3}-3}{12}\ell^2.$$