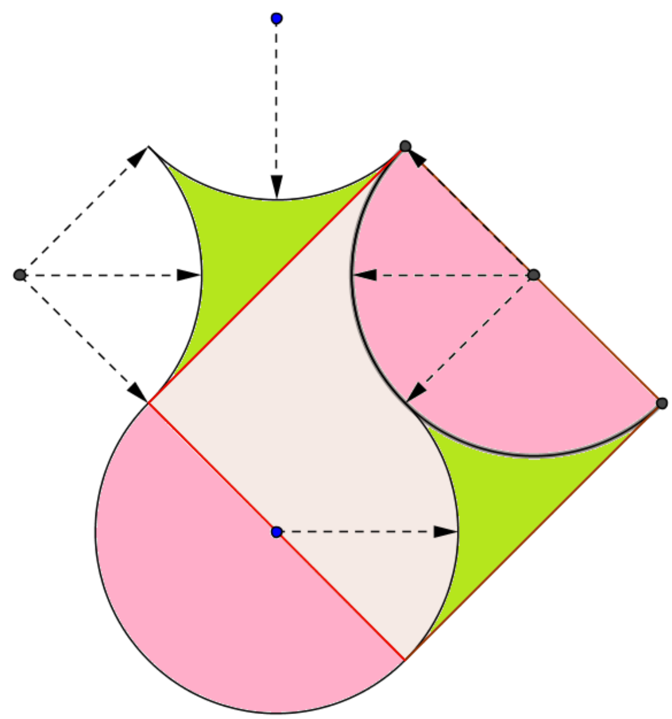
Math Circle at FAU

SOLUTIONS 11/3/2018

1. Cut the jug into 3 pieces by 2 (two) **straight** cuts, and form a square out of the parts. The dashed arrows could be a nuisance, or of some help. You can ignore them, if you wish. We have additional pictures available for experimentation. Scissors and rulers are provided. Look also at the remaining problems.



2. Pepe's car averages 60 miles per gallon of gasoline, Poppy's car averages 40 miles per gallon of gasoline. They both drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

(A) 46 **(B)** 48 **(C)** 50 **(D)** 54 **(E)** 56

Solution. Since the amount driven seems to be unimportant, let's assume both drove 120 miles (to work with integers). Then Pepe's car used up two gallons of gasoline, Poppy's car 3 gallons; between both they used up 5 gallons of fuel. They drove $120 + 120 = 240$ miles. They did 240 miles on 5 gallons, so they were using up fuel at the rate of $240/5 = 48$ miles per gallon. The correct answer is B.
Notice that the combined average is what is called the *harmonic average* of the two rates. If a, b are positive numbers, their harmonic average is

$$\frac{1}{\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)} = \frac{2ab}{a+b},$$

the reciprocal of the arithmetic average of the reciprocals.

3. $\dfrac{2+4+6+\cdots+34}{3+6+9+\cdots+51} = \dfrac{a}{b}$, where a, b are integers with no common divisors other than 1. What are a, b?
Solution.

$$\frac{2+4+6+\cdots+34}{3+6+9+\cdots+51} = \frac{2(1+2+3+\cdots+17)}{3(1+2+3+\cdots+17)} = \frac{2}{3}.$$

4. When Walter drove up to the gasoline pump, he noticed that his gasoline tank was 1/8 full. He purchased 7.5 gallons of gasoline for 22 dollars. With this additional gasoline, his gasoline tank was then 5/8 full. How many gallons of gasoline does his tank holds when it is full?

   Solution. If x is the content in gallons of Walter's tank, then $\frac{1}{8}x + 7.5 = \frac{5}{8}x$. Solving, $x = 15$ gallons.

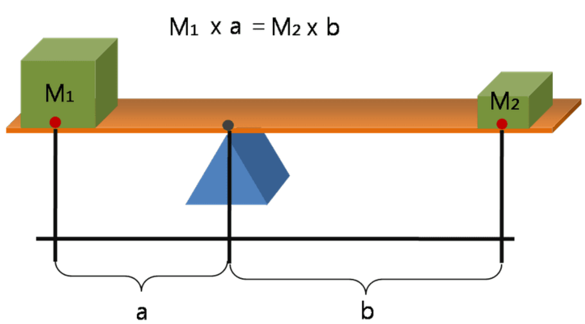
5. A point is chosen at random from within a circular region. What is the probability that the point is closer to the center of the region than it is to the boundary of the region?

   Solution. It might be good to consider a concrete circle, say one of radius 1. Points closer to the center then to the rim are in a circle with the same center, but radius 1/2. The area of that region is $(1/2)^2\pi = \pi/4$.The full circle has area π. The probability we are looking for is $(\pi/4)/\pi = 1/4$.

6. Suppose x is a positive number (not necessarily an integer) Show that $x + \frac{1}{x} = 2$ only if $x = 1$; if $x \neq 1$ then $x + \frac{1}{x} > 2$.

   Solution. Of course $x + \frac{1}{x} = 2$ if $x = 1$. In general, if $x > 0$, we consider $x + \frac{1}{x} - 2$. If we multiply this expression by x we get $x^2 + 1 - 2x = (x-1)^2 > 0$ if $x \neq 1$. (Squares of non zero real numbers are always positive). Dividing next by x, because a positive number divided by a positive number stays positive, we see that $x + \frac{1}{x} - 2 > 0$ or $x + \frac{1}{x} > 2$ if $x \neq 1$.

7. Do you know how levers work? Here is all you need to know. The picture below shows a lever, a board in this case, that can pivot (turn) freely about the point where it is suspended on the triangular shaped support, called a *fulcrum*. Two objects are placed on both ends. For balance, the weight of the object on the left, times its distance to the fulcrum point must equal the weight of the object on the right times its distance to the fulcrum. So a very light object can balance a very heavy one, as long as the light object is sufficiently far away from the fulcrum.



   Here is the problem.

   A farmer, who has come to market with walnuts to sell, finds himself with an accurate 1kg weight and an inaccurate beam balance whose arms are not equal. To weigh out 2 kg of nuts to a customer, the farmer first puts the weight in the right pan of the balance and adds nuts in the left until they balance. He places these nuts in a bag. Next he balances the weight in the left pan with nuts in the right, adding these to the bag. **(a)** Show (explain) that the resulting bag of nuts weights more than 2 kg. **(b)** How can the farmer weigh accurately 2 kg of nuts?

   Note The previous problem could play a role here.

Solution. Suppose that L and R are the lengths of the left and right arms of the balance, respectively. Suppose w is the weight of the first batch of nuts, v the weight of the second batch. By the laws of the lever, $Lw = R$ and $L = Rv$. Solving for w, v, we get $w = R/L, v = L/R$ so that $w + v = (R/L) + (L/R)$. Let $x = R/L$ Since $R \neq L$, $x \neq 1$ and $v + w = x + (1/x) > 2$. This takes care of part (a). For an accurate weight, the farmer can place the 1 kg weight in the right pan, pour nuts until balance is achieved into the left pan. The nuts in the left pan now balance exactly 1kg in the right. The farmer can now remove the weight from the right pan and pour into it nuts until balance is again achieved. He now has an exact 1 kg of nuts in the right pan. Take it out, put it in the bag, and repeat to get a second kilogram.

8. **(a)** Blindfolded you enter a room on which there are two tables. The table on your left has a layer of nickels, one nickel deep, and the one on your right is empty. You are wearing gloves, so you can't tell what coins are heads up or tails up, but you know that 31 coins are tails up, the rest are heads up. You can transfer any number of coins from one table to the other and, at the same time, flip any desired number of coins. Can you arrange things so that, possibly after several transfers, the number of coins that are tails up is the same on both tables? **(b)** What if the initial situation is that there are nickels on both tables, with 31 of those on the left table and 8 of those on the right table tails up?

   Solution. For part (a), transfer 31 coins from the left table to the right table, flipping all of them. Of these 31 coins there could have been m that were tails up and n that were heads up. That leaves $31 - m = n$ coins tails up on the left table. Flipping the 31 coins you transferred changes the n heads to tails. So now we have n coins tails up on both tables. To give a concrete example, if of the 31 transferred coins, 3 were tails up, that leaves 28 coins tails up on the left table. On the right table, before flipping, we have 28 heads up, 3 tails up; on flipping both tables have 28 tails up.
   For part (b) first transfer all coins on the right table to the left table, without flipping. The right table is now empty, the left table has 39 coins that are tails up. Now transfer 39 coins from the left to the right table and flip them all. Alternatively, you can just transfers $23 = 31 - 8$ coins from the left to the right table, flipping them.

9. A wire is cut into two pieces, of lengths a and b. The piece of length a is bent to form an equilateral triangle. The piece of length b to form a regular hexagon. The triangle and the hexagon have the same area. What is the ratio $\frac{a}{b}$?

   Solution. The area of an equilateral triangle of sides of length L is $\frac{\sqrt{3}}{4}L^2$. The triangle made out of the wire of length a has sides $L = a/3$, so area $\frac{\sqrt{3}}{36}a^2$. The hexagon can be decomposed into 6 equilateral triangles of sides of length $b/6$, so its area is $6\frac{\sqrt{3}}{4}\left(\frac{b}{6}\right)^2 = \frac{\sqrt{3}}{24}b^2$. Equating the two areas,

$$\frac{\sqrt{3}}{36}a^2 = \frac{\sqrt{3}}{24}b^2,$$

   so $a^2/b^2 = 36/24 = 3/2$. Taking square roots, we get $a/b = \sqrt{3}/\sqrt{2}$.

10. Sarah wants to arrange the mini candy bars she got at Halloween. When she divides them into bunches of two each, she has one bar left over. Grouping them into bunches of three bars, she has one bar left over. The same happened when she divided the bars into bunches of 4 bars, and bunches of 5 bars. If it is known that Sarah had less than 100 mini candy bars, how many candy bars did Sarah have?

    Solution. There is a sophisticated way of doing this, but we can go by trial and error (100 is not a terribly large number). Having one left over, means she had an odd number of bars to start with. That she also had one left over when dividing into bunches of three, means the original number had to be a multiple of 3 plus one. It also had to be a multiple of 5 plus 1 (We leave 4 to the end) It is not too horrible to list all odd numbers that exceed a multiple of 3 and 5 by 1. They are (1 is included because 0 is a multiple of every number): 1, 31, 61, 91. Of these only 1 and 61 exceed a multiple of 4 by one unit. We may assume she had more than one candy bar and was not ghost arranging bars into empty piles, so the answer is 61 bars.

11. We are going to look here for integers with the following curious property. If the integer has n digits, then the last n digits of its square equal the integer. For example, 76 has this property because $76^2 = 76 \times 76 = 5775$

and the last two digits of 5776 spell 76. Another example is 25; $25^2 - 6\mathbf{25}$. Let us give a name to the numbers we are looking for; sometimes they are called *automorphic numbers*. The only one digit automorphic numbers are 1, 5, and 6. **If you don't understand or agree with this last sentence, talk with one of the organizers, ask for a better explanation!**

Here is the problem.

(a) Show that 25 and 76 are the only two digit automorphic numbers.

(b) Find ALL three digit automorphic numbers.

(c) True or false, and why: If a number of n digits is automorphic, then the last n digits of its cube also spell out the number.

Solution. We will use the fact that when we add, subtract, or multiply numbers, the last digit of the result is the same as the last digit we would get operating only on the last digit. The same is true for the last two digits, the last three digits, etc. For example, consider $581 \times 334 = 194,054$. Notice that $1 \times 4 = 4$, the last digit of the product. If we take the two last digits of the operands and multiply them we get $81 \times 34 = 2754$; the last two digits are 54, same as of the original product. Another thing that we will use is that if two numbers have the same last digit, their difference is divisible by 10, if the same two last digits, their difference is a multiple of 100, if the same last three digits, their difference is a multiple of 1000. Given this, we conclude that if a is an automorphic number, then its last digit must be automorphic, its last two digits have to be automorphic, and so forth. Since 1, 5, 6 are the only one digit numbers that work, every automorphic number has to end either in 1, 5, or 6. Suppose we now have a two digit automorphic number. Suppose it ends in 1, so it looks like $a1 = 10a + 1$. Squaring $(10a + 1)^2 = 100a^2 + 20a + 1$. If this is to have $a1$ as the last two digits, we must have that

$$(100a^2 + 20a + 1) - (10a + 1) = 100a^2 + 10a$$

is a multiple of 100. This forces $10a$ to be a multiple of 100, possible only if $a = 0$. But then $a1 = 1$ does not have two digits. Next we try $a5 = 10a + 5$. Now

$$(10a + 5)^2 - (10a + 5) = 100a^2 + 100a + 25 - 10a - 5 = 100a + 100a + 20 - 10a,$$

and we should have $20 - 10a$ a multiple of 100. This only happens if $a = 2$, so $20 - 10a = 0$. It follows that 25 is the only automorphic two digit number ending in 5. The analysis for numbers ending in 6 is similar. We should have

$$(10a + 6)^2 - 10a - 6 = 100a^2 + 110a + 30$$

divisible by 100, thus $110a + 30 = 100a + 10a + 30$ divisible by 100, and one sees nothing but $a = 7$ works. This tales care of part a. To get all three digit numbers of this type we now know they have to end in 25 or 76. We proceed more or less the same as before, except that where we wondered about divisibility by 100, we now wonder about divisibility by 1000.

Ending in 25: We ask for what digit a is $(100a + 25)^2 - (100a + 25)$ divisible by 1000. Now

$$(100a + 25)^2 - (100a + 25) = 10000a^2 - 5000a + 625 - 100a - 25$$

has to be a multiple of 1000, which is equivalent to $625 - 100a - 25 = 600 - 100a$ being a multiple of 1000, only possible if it is 0, or $a = 6$. So 625 is the only automorphic three digit number ending in 25. Ending in 76: To se if there is one ending in 76 we do essentially the same thing. I'll be briefer this time.

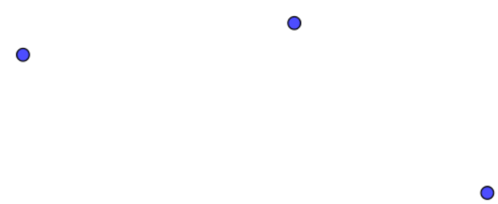
$$(100a + 76)^2 - (100a + 76) = 10000a^2 + 15100a + 5700.$$

We need to find a, if any, such that $15100a + 5700 = 15000a + 5000 + 700 + 100a$ is a multiple of 1000. It is easy to see that only 3 works. The only such number is 376. In sum, the only 3 digit automorphic numbers are 625 and 376.

Concerning part c, it is true. A number of n digits x is automorphic if and only if 10^n divides $x^2 - x$. Because $x^3 - x = x(x^2 - x) + x^2 - x$, 10^n also divides $x^3 - x$, and the last n digits of x^3 coincide with the digits of x.

12.

13. Frida drew a triangle, marked the mid points of the three sides. Then she erased the sides, leaving only the midpoints, as seen in the figure below. She then challenged her friend Barney to reconstitute the triangle, using only a straightedge[1] and a compass. Can you help Barney do it?



Given three points, how can you construct, using only a compass and a straightedge, a triangle so that the given three points are the midpoints of the sides? Once you figure it out, please explain how to do it.

Solution. The following property of triangles plays the main role: Suppose P, Q are the midpoints of sides AC and BC of triangle ABC. Then PQ is parallel to AB. **Can you prove this?**
In view of this property what we need to do with a compass and straightedge is for each one of the given points, draw a line through it parallel to the line determined by the other two points. These lines contain the sides of the triangle; their intersections are the vertices. The next question is, of course, how do we draw a line parallel to a given line through a given point. There are many ways of doing this; the way we use is perhaps shorter than others. To draw a line through a given point A parallel to the line through points B, C we proceed as follows:

- Find the midpoint of the segment AC, call it X. This can be done by drawing a circle of center A, radius AC, one of center C, radius AC. These circles intersect at points P, Q on opposite sides of the segment AC; the segment PQ intersects AC exactly at its midpoint. (Fig. 1 below)
- Draw the line through B and X. (Fig. 2)
- With center at X draw a circle of radius XB. It intersects the line through BX at B and at a point R.(Fig 3)
- The line determined by AR is the line parallel to BC through A. (Fig 4)

The figures below show the procedure using the given points. I named the top one of the three given points A, the other two B, C. Auxiliary lines and circles are drawn with dotted lines and erased once not needed anymore. Once you have the first line parallel to the line through two of the points through the third point, repeat for the other given points.
Can you explain why the line AR is parallel to BC?

[1] A straightedge is an unmarked ruler. You can only use it to draw straight lines, you can't measure with it.

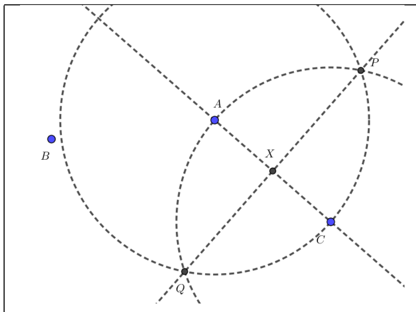


Figure 1

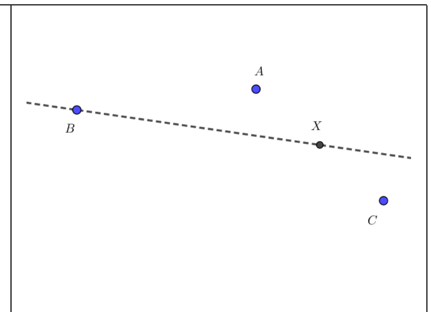


Figure 2

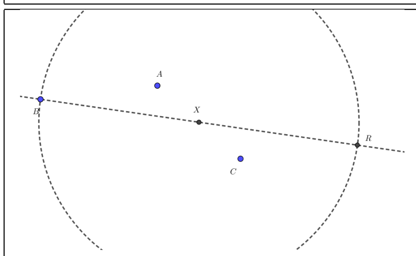


Figure 2

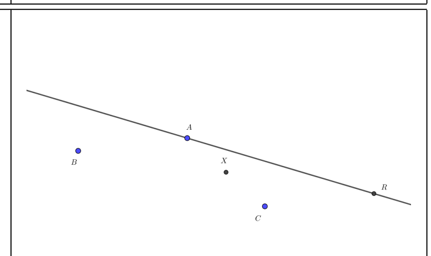
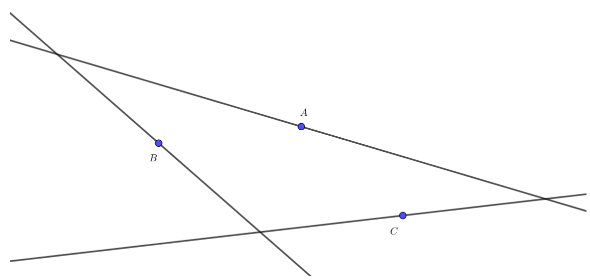


Figure 3

We now have drawn a line containing the side of the triangle of which A is the midpoint. We repeat the process with the other two points to get:



Mark the points where these three lines intersect, they are the vertices of the triangle we are looking for.