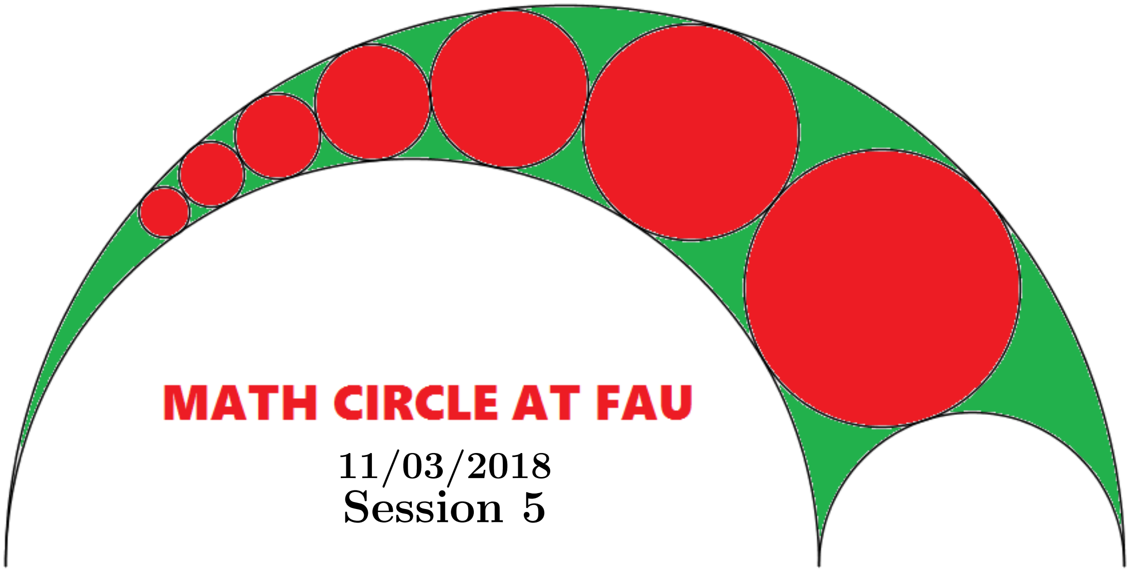


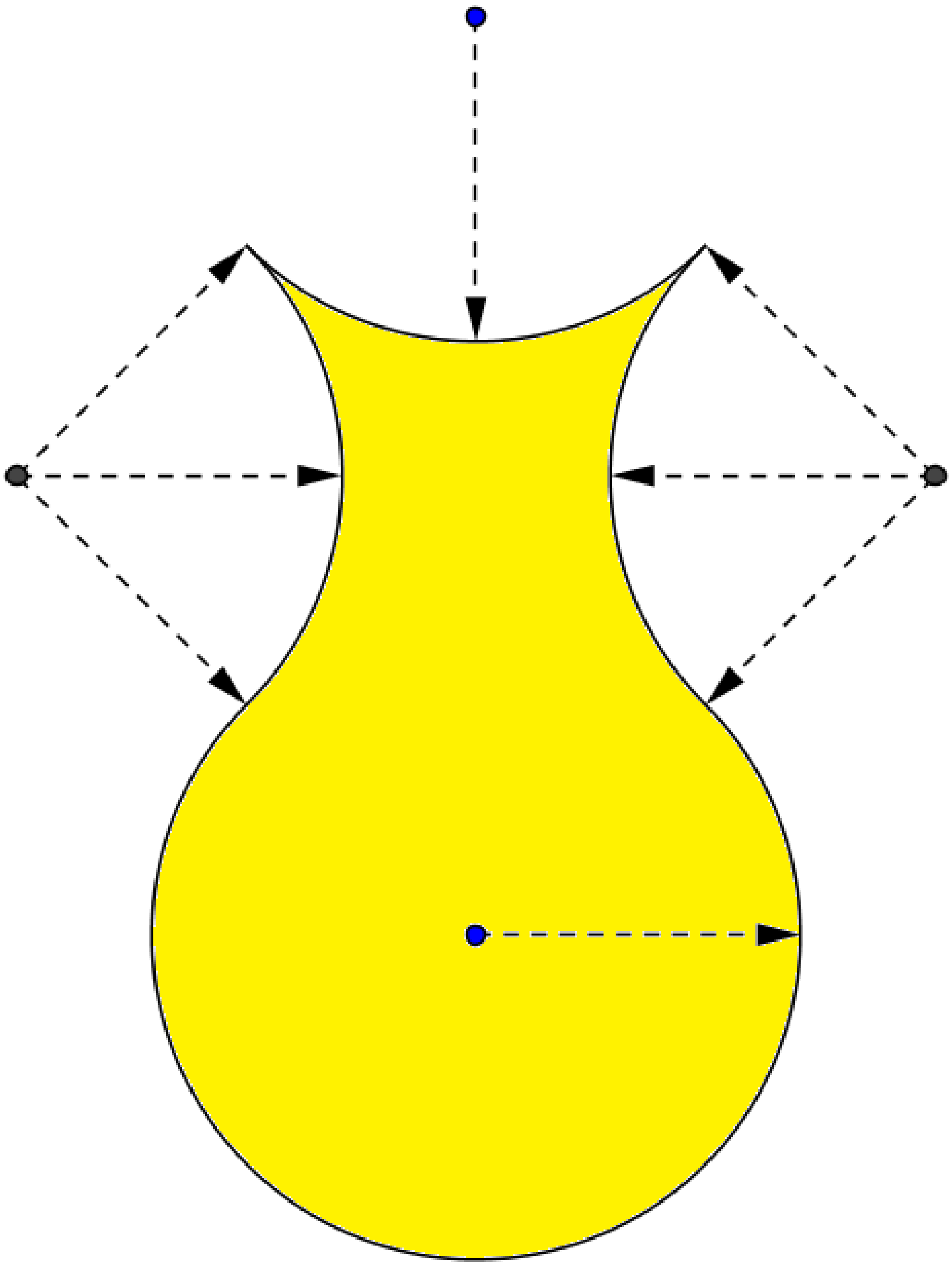
MATH CIRCLE AT FAU

11/03/2018
Session 5

RULES

- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.
- Except in case of an emergency, all smart devices are strictly forbidden. No smart phones, smart watches, or time machines. Only smart people.
- Problems might get harder toward the end. **Please don't feel frustrated if you can't do them.** Still, you can ask an organizer for explanations; maybe with a few hints you can work the problem.

1. Cut the jug into 3 pieces by 2 (two) **straight** cuts, and form a square out of the parts. The dashed arrows could be a nuisance, or of some help. You can ignore them, if you wish. We have additional pictures available for experimentation. Scissors and rulers are provided. Look also at the remaining problems.



2. Pepe's car averages 60 miles per gallon of gasoline, Poppy's car averages 40 miles per gallon of gasoline. They both drive the same number of miles. What is the cars' combined rate of miles per gallon of gasoline?

(A) 46 **(B)** 48 **(C)** 50 **(D)** 54 **(E)** 56

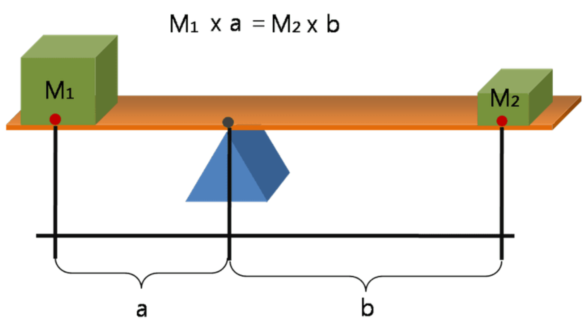
3. $\dfrac{2+4+6+\cdots+34}{3+6+9+\cdots+51} = \dfrac{a}{b}$, where a, b are integers with no common divisors other than 1. What are a, b?

4. When Walter drove up to the gasoline pump, he noticed that his gasoline tank was 1/8 full. He purchased 7.5 gallons of gasoline for 22 dollars. With this additional gasoline, his gasoline tank was then 5/8 full. How many gallons of gasoline does his tank holds when it is full?

5. A point is chosen at random from within a circular region. What is the probability that the point is closer to the center of the region than it is to the boundary of the region?

6. Suppose x is a positive number (not necessarily an integer) Show that $x + \dfrac{1}{x} = 2$ only if $x = 1$; if $x \neq 1$ then $x + \dfrac{1}{x} > 2$.

7. Do you know how levers work? Here is all you need to know. The picture below shows a lever, a board in this case, that can pivot (turn) freely about the point where it is suspended on the triangular shaped support, called a *fulcrum*. Two objects are placed on both ends. For balance, the weight of the object on the left, times its distance to the fulcrum point must equal the weight of the object on the right times its distance to the fulcrum. So a very light object can balance a very heavy one, as long as the light object is sufficiently far away from the fulcrum.



Here is the problem:

A farmer, who has come to market with walnuts to sell, finds himself with an accurate 1kg weight and an inaccurate beam balance whose arms are not equal. To weigh out 2 kg of nuts to a customer, the farmer first puts the weight in the right pan of the balance and adds nuts in the left until they balance. He places these nuts in a bag. Next he balances the weight in the left pan with nuts in the right, adding these to the bag. **(a)** Show (explain) that the resulting bag of nuts weights more than 2 kg. **(b)** How can the farmer weigh accurately 2 kg of nuts?

Note The previous problem could play a role here.

8. **(a)** Blindfolded you enter a room on which there are two tables. The table on your left has a layer of nickels, one nickel deep, and the one on your right is empty. You are wearing gloves, so you can't tell what coins are heads up or tails up, but you know that 31 coins are tails up, the rest are heads up. You can transfer any number of coins from one table to the other and, at the same time, flip any desired number of coins. Can you arrange things so that, possibly after several transfers, the number of coins that are tails up is the same on both tables? **(b)** What if the initial situation is that there are nickels on both tables, with 31 of those on the left table and 8 of those on the right table tails up?

9. A wire is cut into two pieces, of lengths a and b. The piece of length a is bent to form an equilateral triangle. The piece of length b to form a regular hexagon. The triangle and the hexagon have the same area. What is the ratio $\frac{a}{b}$?

10. Sarah wants to arrange the mini candy bars she got at Halloween. When she divides them into bunches of two each, she has one bar left over. Grouping them into bunches of three bars, she has one bar left over. The same happened when she divided the bars into bunches of 4 bars, and bunches of 5 bars. If it is known that Sarah had less than 100 mini candy bars, how many candy bars did Sarah have?

11. We are going to look here for integers with the following curious property. If the integer has n digits, then the last n digits of its square equal the integer. For example, 76 has this property because $76^2 = 76 \times 76 = 5775$ and the last two digits of 5776 spell 76. Another example is 25; $25^2 - 6\mathbf{25}$. Let us give a name to the numbers we are looking for; sometimes they are called *automorphic numbers*. The only one digit automorphic numbers are 1, 5, and 6. **If you don't understand or agree with this last sentence, talk with one of the organizers, ask for a better explanation!**

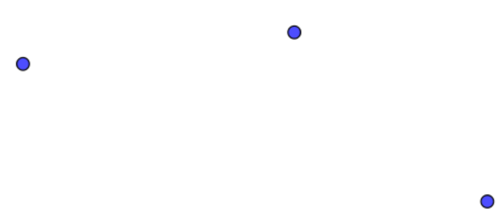
    Here is the problem.

    (a) Show that 25 and 76 are the only two digit automorphic numbers.

    (b) Find ALL three digit automorphic numbers.

    (c) True or false, and why: If a number of n digits is automorphic, then the last n digits of its cube also spell out the number.

12. Frida drew a triangle, marked the mid points of the three sides. Then she erased the sides, leaving only the midpoints, as seen in the figure below. She then challenged her friend Barney to reconstitute the triangle, using only a straightedge[1] and a compass. Can you help Barney do it?



Given three points, how can you construct, using only a compass and a straightedge, a triangle so that the given three points are the midpoints of the sides? Once you figure it out, please explain how to do it.

[1] A straightedge is an unmarked ruler. You can only use it to draw straight lines, you can't measure with it.