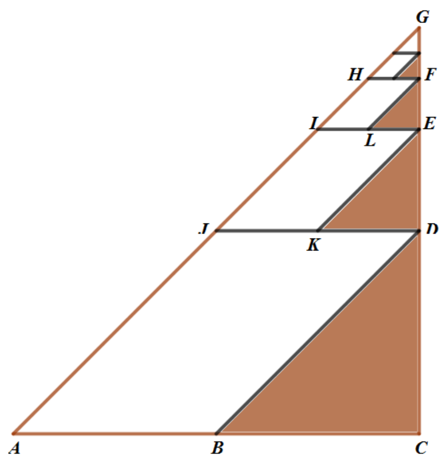
Math Circle at FAU
10/27/2018

SOLUTIONS

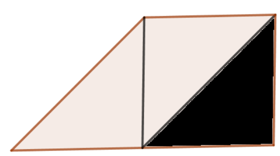
1. At the grocery store last week, small boxes of facial tissue were priced at 4 boxes for \$5. This week they are on sale at 5 boxes for \$4. Find the percent decrease in the price per box during the sale. (AJHSME)

   Solution. The original price per box was \$ $5/4 =$ \$ 1.25. The price goes down to \$ $4/5 =$\$0.80, a decrease of \$ 0.45 Since we are asking for the percentage of the decrease we have to figure out what percentage 0.45 is of 1.25. Now $0.45/1.25 = 0.36$. The answer is 36%.

2. Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and $AC = CG = 6$, then the integer nearest to the total area of the shaded triangles is ? (AMC 8)

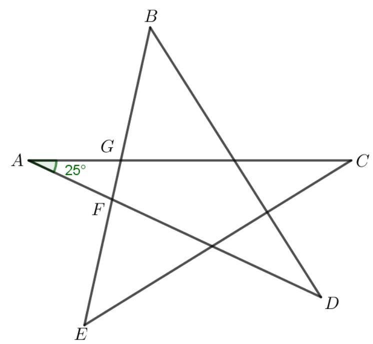
   Solution. Consider the portion of triangle ACG between the base and the top of one of the shaded triangles. The picture looks like the figure on the left, a trapezoid.

   Drawing a vertical line from the midpoint of the base of this trapezoid, as shown on the right, we see that the trapezoid divides into three congruent triangles of which one is the shaded one. So the area of the shaded triangle is one third the area of the trapezoid. Since the original triangle is made up of an infinity of these stacked trapezoids, after one hundred steps we should be quite close to having shaded one third of the area of the triangle. The area of the triangle is $\frac{1}{2}(6 \times 6) = 18$, so after 100 steps we should be quite close to having an area of $18/3 = 6$. So the nearest integer to the area is 6.

   Here is a more precise, and perhaps easier way, of coming to the same conclusion. The area of the first shaded triangle is $\frac{1}{2}(3 \times 3) = 4.5$; the second one has area $\frac{1}{2}(1.5 \times 1.5) = 1.125$, the third shaded rectangle has area $\frac{1}{2}(0.75 \times 0.75) = 0.28125$. The areas of the first three shaded triangles add up to 5.90625; much closer to 6 than to 5, and an underestimate. If we add to this the are of triangle HFG we'll have an overestimate; the area of $\triangle HFG$ is 0.28125, same as the area of the l=third shaded triangle; adding it to the area of the first three shaded triangles we get: 6.1875. The area must be somewhere in between; the closest integer is 6

3. In the star shaped figure, the angle at A measures $25°$ and $\angle AFG = \angle AGF$. Find $\angle B + \angle D$. (AMC 8)



Solution. $\angle B + \angle D = \angle DFE = \angle AFG$; since $\angle AFG = \angle AGF$ and $\angle AFG + \angle AGF + 25° = 180°$, $\angle AFG = \frac{1}{2}(180 - 25)° = 75.5°$, so $\angle B + \angle D = 75.5° = 75°30'$.

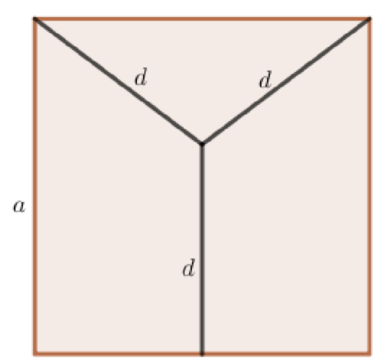
4. Show that if the three digit number abc is divisible by 7, then so is $cba - (c - a)$. For example, 133 is divisible by 7, and so is $331 - (3 - 1) = 331 - 2 = 329$. Here is another example: 623 is divisible by 7 and so is $326 - (3 - 6) = 326 - (-3) = 326 + 3 = 329$.

   Solution. Let $x = abc$, $y = cba - (c - a)$. By abc we really mean $100a + 10b + c$; by cba we mean $100c + 10b + a$. Then $y = cba - (c - a) = 100c + 10b + a - c + a = 99c + 10b + 2a$. It follows that $x - y = 98(a - c)$. Since 98 is divisible by 7, so is $98(a - c)$. It follows that y is divisible by 7 if and only if x is divisible by 7.

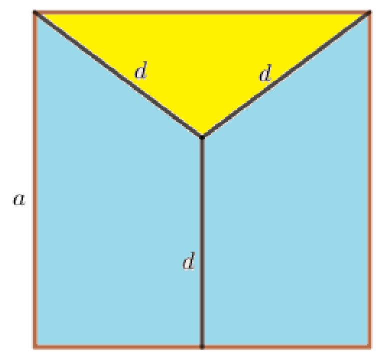
5. A man living in the suburbs and working in the city returns every day by a train that arrives at his suburban station at exactly 5 p.m. His butler picks him up at the station to drive him home; the butler has arranged things so he arrives at exactly 5 p.m. at the station. The route the butler takes to the station is the same as he takes to drive home. One day the man takes an earlier train, arriving at the station at 4. He decides to start walking home, along the route taken always by the butler. Somewhere along the route he meets the butler; he jumps into the car and the butler drives him home arriving 10 minutes earlier than usual. How long had the man be walking before being picked up by the butler? Assume that getting in and out of a car, turning the car around, everything but the driving (and walking) is instantaneous.

   Solution. The round trip took 10 minutes less than usual, so the car trip up to the station took 5 minutes less than usual. In five more minutes the butler would have been at the station, so it was 4:55 when he met the man, meaning the man must have walked for 55 minutes.

6. A point P inside a square of side length a is at the same distance d from two consecutive vertices and from the side opposite the two vertices. What is a/d?



   Solution. Consider the picture, with some colors.



We equate the area of the square, which is a^2 to the sum of the area of the yellow triangle and the areas of the two congruent light blue trapezoids. The area of the yellow triangle can be obtained by Heron's formula. The semi perimeter of the triangle is $(2d+a)/2 = d + \frac{a}{2}$,so Heron's formula gives the area as

$$\sqrt{\left(d+\frac{a}{2}\right)\left(d-\frac{a}{2}\right)\frac{a}{2}\cdot\frac{a}{2}} = \frac{a}{2}\sqrt{d^2-\frac{a^2}{4}}.$$

Alternatively, one can use Pythagoras' Theorem to see that if we take the top side of the square as the base of the triangle, then the altitude works out to $\sqrt{d^2-\frac{a^2}{4}}$ and end with the same expression for the area. Each blue trapezoid has bases of length a and d, has altitude $a/2$, so each one has an area of

$$\frac{1}{2}(a+d)\frac{a}{2}.$$

Equating as mentioned above we get

$$a^2 = \frac{a}{2}\sqrt{d^2-\frac{a^2}{4}} + 2\left(\frac{1}{2}(a+d)\frac{a}{2}\right) = \frac{a}{2}\left(\sqrt{d^2-\frac{a^2}{4}}+a+d\right).$$

Multiplying by $2/a$ and rearranging we get $a-d = \sqrt{d^2-\frac{a^2}{4}}$. Squaring we get

$$a^2-2ad+d^2 = d^2-\frac{a^2}{4},$$

from which we get $d = 5a/8$. The answer is $\frac{a}{d} = \frac{8}{5}$.

7. Three friends, Amy, Bertie and Chippy, have some money and not only are they very good friends, but they also are very generous. So Amy decides to give Bertie and Chippy some of her money, after which Bertie and Chippy have twice as much money as before. Then Bertie gives money to Amy and Chippy, doubling their amount of Money. Finally, Chippy gives away money to Amy and Bertie, doubling their amounts. If Chippy started with \$ 72 and at the end has again \$ 72, what is the total amount of money the three friends have? (AMC 8)

   Solution. We can concentrate on Chippy who starts with 36 dollars, after the first round he has 72 dollars, and after the second round 144 dollars. Since he will end with 36 dollars after the next round he must be giving away 108 dollars. This doubles Amy's and Bertie's amounts, so they must have at the end of the second round a combined total of \$ 108. If we add to this the \$ 144 of Chippy at the end of this round, we see that the total amount of money of all three is $108 + 144 = 252$ dollars.

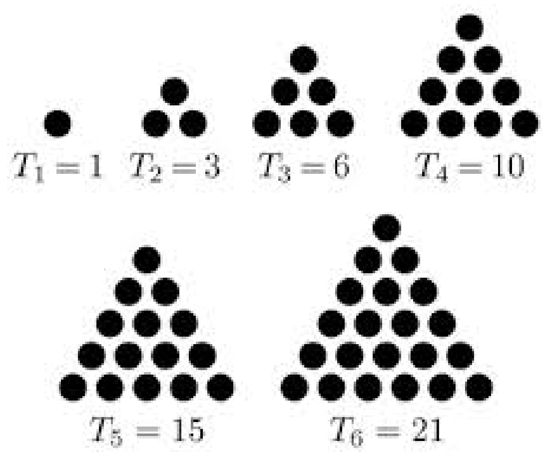
8. Find the area of the trapezoid pictured below where $AB = 23$, $BC = 8$, $CD = 13$ and $DA = 6$.

Solution. The area of a trapezoid is given by $A = \frac{1}{2}(a+b)h$, where a, b are the bases and h is the height. In our case $a = 23$ and $b = 13$; all we need is h. There are many ways of getting h, but here is a very simple one. Here is the picture of the trapezoid with two triangles marked off.

Slide the triangle on the left until it bumps into the triangle on the right. We then get a triangle of sides of length 6, 8, 10. We might recognize this as a right triangle of are $\frac{1}{2}(8 \times 6) = 24$. Taking thehypotenuse as base,theheight is the same as the height of the trapezoid. Its area is also $\frac{1}{2}(10 \times h)$. Equating the two expressions for the area, we can solve to get $h = 24/5$. The area of the trapezoid is thus

$$\frac{1}{2}(23+13) \times \frac{24}{5} = \frac{432}{5}.$$

9. *Triangular numbers* are numbers that can be represented by triangular shapes made out of dots. The first few are



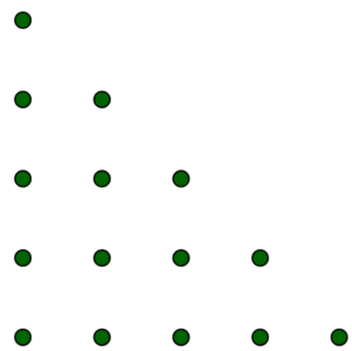
(a) Show, explain $1 + 2 + \cdots + n = T_n$.

(b) Find a simple formula for T_n.

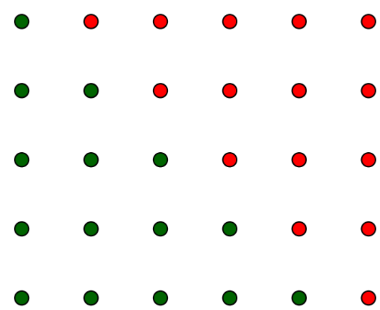
Solution.

(a) Notice that one gets each triangular number from the previous one by adding a row of dots containing one dot more than the bottom row of the previous number. So $T_1 = 1$, $T_2 = 1+2$, $T_3 = T_2+3 = 1+2+3$, and so forth.

(b) Using two copies of T_n you can make an $n \times (n+1)$ rectangle of dots. Thus $T_n = \dfrac{n(n+1)}{2}$. The procedure is illustrated for T_5. We write $T5$ in the form

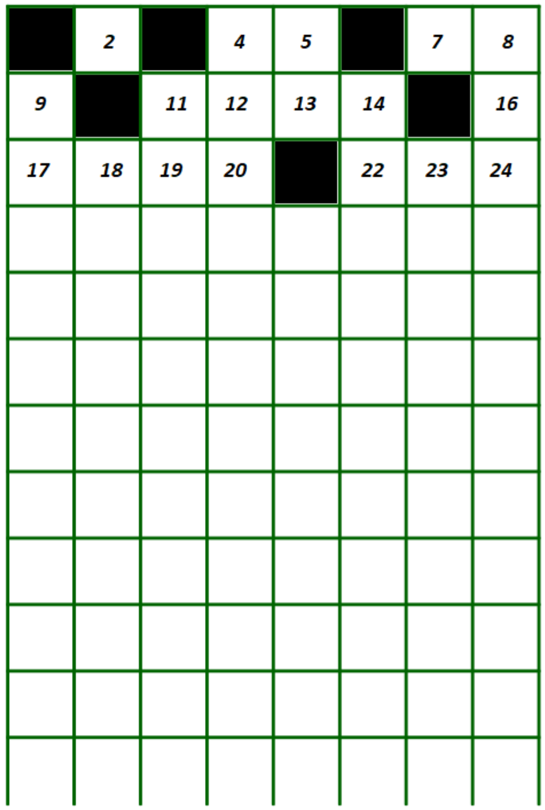


Next we rotate another copy of $T5$ and add it to the first as shown in the picture below; the green dots are the first copy, the red dots the new copy.



The second picture shows that $2T_5 = T_5 + T_5 = 5 \times 6$.

10. A rectangular board of 8 columns has squares numbered beginning in the upper left corner and moving left to right so row one is numbered 1 through 8, row two is 9 through 16, and so on. A student shades square 1, then skips one square and shades square 3, skips two squares and shades square 6, skips 3 squares and shades square 10, and continues in this way until there is at least one shaded square in each column. What is the number of the shaded square that first achieves this result? (AJHSME)



Solution. The numbers of the squares getting shaded are 1, 3, 6, 10, 15, 21, etc. These are the triangular numbers; triangular numbers have the form $n(n+1)/2$ for $n = 1, 2, 3, \ldots$. Of the eight columns of the board, column 1 is filled immediately, so are columns 3 and 6. If we look at the picture, the only columns left without a shaded square are columns 4 and 8. The numbers in column 4 are $4, 12, 20, 28, \ldots$; when is any of these entries a triangular number? We need to find n so $n(n+1)/2 = 4 + 8k$, with k as small as possible. Can we have $n(n+1)/2 = 4 + 8 = 12$; that is $n(n+1) = 24$? It is easy to see the answer is no. So we try, $n(n+1)/2 = 4 + 8 \cdot 2 = 20$; that is $n(n+1) = 40$. We again get a negative answer. The next case leads to $n(n+1) = 72$, and this works. Trying to keep it small, we take $n+1 = 9, n = 8$, so $n(n+1)/2 = 36$. So column 4 will get square 36 shaded. Next to column 8. We need n so $n(n+1)/2$ is a multiple of 8; equivalently $n(n+1)$ is a multiple of 16. Of $n, n+1$ one is even the other one odd. So the only way that $n(n+1)$ can be a multiple of 16 is if either n or $n+1$ is a multiple of 16. The smallest way this can happen is if $n+1 = 16$, so $n = 15$ and $n(n+1)/2 = 120$. The answer is that column 8, and thus all columns, will contain a shaded square once we shade square number 120.

11. This is an almost Fibonacci problem. There used to be a time when people wrote letters to friends and family, instead of texting or email. In those far, far gone days, a person, lets call him Armand, wrote 7 letters to 7 friends. Armand had previously addressed 7 envelopes for the letters written. In how many ways can Armand place **every** single letter into the wrong envelope?

    (As with so many problems one should start with easy cases. For example if there is one letter, one envelope, there are 0 ways of making a mistake. If there are two letters (lets call them L_1 and L_2) and two envelopes (E_1 and E_2) then there is one way; L_1 into E_2 and L_2 into E_1. And so it goes.)

    Solution. Let s_n be the number of ways one can place n letters into n envelopes so that **no** letter is in the right envelope. We already saw that $s_1 = 0, s_2 = 1$. Suppose now we have 3 letters and envelopes. We then have 2 ways of putting letter 3 into a wrong envelope. Now we have to be careful. Envelope 3 is free, so is the envelope of the second or third letter. There are no choices left. If, to give an example, we placed letter 3 into envelope 2, we now have to place letter 1 into envelope 2 and letter 2 into envelope 1. So $s_3 = 2$. It may be time to see if we can get a formula. If we have n letters and n envelopes, letter n can be messed up in $n-1$ ways; it can go into envelopes $1, 2, \ldots, n-1$. We now have two cases, (a) letter n goes into envelope

k (k one of $1, \dots, n-1$) and letter k goes to envelope n. There are $n-1$ ways this can happen. For each one of these ways we are left with $n-2$ letters and their envelopes, allowing us to mess up in s_{n-2} ways. That's a total of $(n-1)s_{n-2}$ ways. Case (b), letter n goes into an envelope k, but letter k does not go into envelope n. We can now declare letter k a temporary letter n and we are left with $n-1$ letters and their corresponding envelopes. So we have $n-1$ envelopes into which letter n can go and then s_{n-1} possibilities for each one of these $n-1$ places where letter n can go; a total of $(n-1)s_{n-1}$ different ways to mess up. To illustrate a bit further a bit more, say $n = 5$ and we look at the case in which letter 5 goes to envelope 3. Now letter 3 can't go into envelope 5 (otherwise we are in case (a)), so we relabel it, call it letter five, perhaps in Roman numerals to not confuse it with the original letter 5. So our remaining letters are 1, 2, 4, V, and envelopes 1, 2, 3, 5. There are s_4 ways to mess up totally.
Combining the ways we can mess up in case (a) and case (b), we get the formula

$$s_n = (n-1)(s_{n-2} + s_{n-1}).$$

So now we have

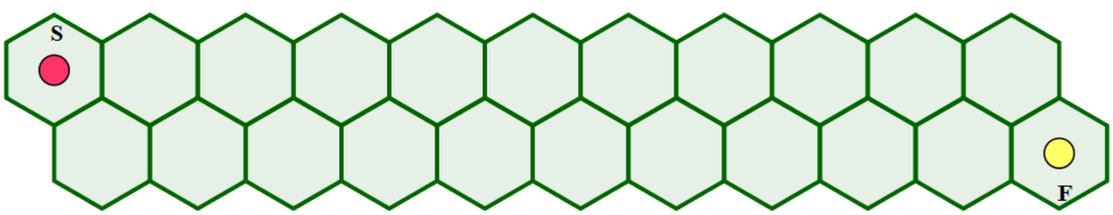
$$\begin{aligned}
s_1 &= 0 \\
s_2 &= 1 \\
s_3 &= 2(s_1 + s_2) = 2 \\
s_4 &= 3(s_2 + s_3) = 9 \\
s_5 &= 4(s_3 + s_2) = 44 \\
s_6 &= 5(s_4 + s_5) = 265 \\
s_7 &= 6(s_5 + s_6) = 1854
\end{aligned}$$

The answer is in 1854 different ways.

The Fibonacci Section

Some of you have perhaps solved last session's "Buzzy the Bee" problem. As a reminder, here it is, with a solution.

Buzzy the Bee starting at cell S with the red dot at the center wants to get to the cell F, with the yellow dot at the center.



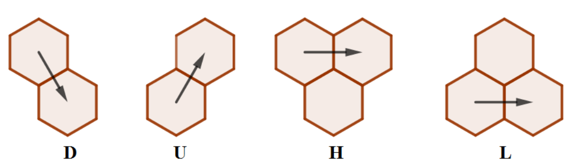
Each step must move to the right and to a neighboring cell. There are four possible types of steps:

D = downward diagonal.

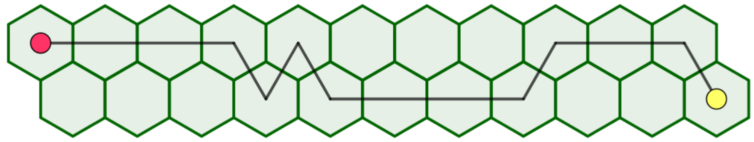
U = upward diagonal.

H = high horizontal step.

L = low horizontal step.



For example, one allowed route is the one shown below:



How many different routes are there from the cell with the red dot to the cell with the yellow dot?

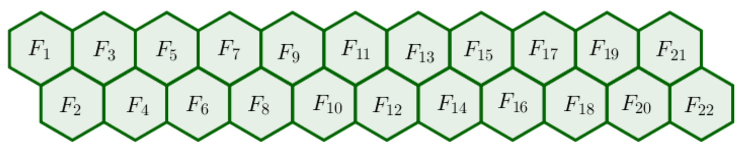
Hint: Starting at the cell with the red dot, enter in each cell the number of different ways it can be reached from the cell with the red dot. It might suggest something.

If you followed the suggestion at the end of the problem, you would discover that the number of different ways to get to the cell n positions to the right of the cell with the red dot was F_n, the n-th Fibonacci number. The Fibonacci numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots;$$

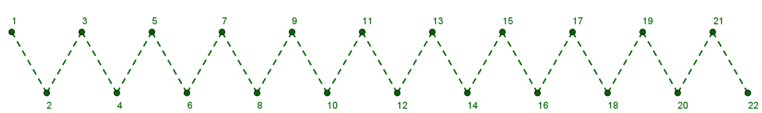
it starts with 1, 1, and from then on every number is the sum of the two preceding ones. The so called recursive definition is $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ once $n \geq 3$.

Your honeycomb then looks like

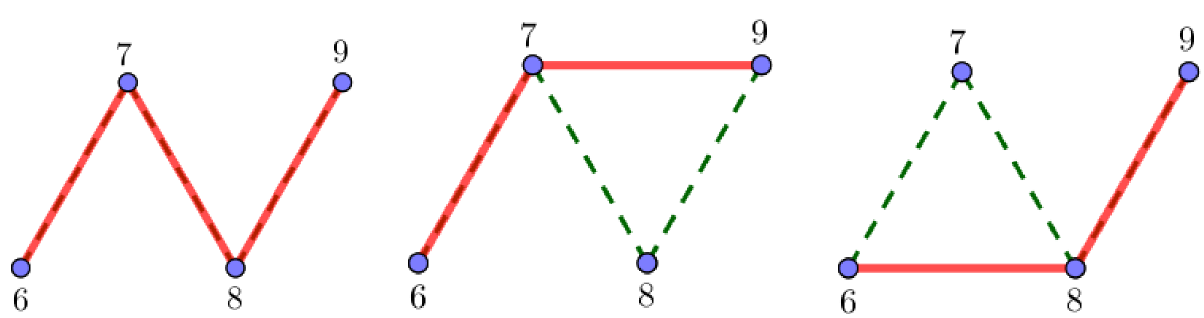


The answer to the question is $F_22 = 17,711$.

The remaining problems build upon this counting of routes. However, the cells are a bit distracting. It's easier perhaps to replace them with dots, and just number the vertices, so the picture looks like



What the argument used to solve the buzzy the bee problem shows is that starting at any vertex, there are exactly F_{k+1} routes to a vertex k positions to the right of it. In other words, if $1 \leq m < n$ then there are F_{n-m+1} different routes from vertex m to vertex n. To give a very simple example the routes from vertex 6 to vertex 9 are:



Because $9 - 6 + 1 = 4$, there are $F_4 = 3$ different routes.

Here are the problems.

1. By counting routes show that if $1 \leq m < n$ then

$$F_n = F_m \cdot F_{n-m+1} + F_{n-m} \cdot F_{m-1}.$$

For example, with $n = 21$, $m = 12$ so that $n - m = 9$, $n - m + 1 = 10$ and $m - 1 = 11$, the formula says

$$F_{21} = F_12 \cdot F_{10} + F_9 \cdot F_{11} = 144 \cdot 55 + 34 \cdot 89 = 10,946.$$

Solution. A route ending at vertex n either goes through vertex m or it doesn't. There are F_{n-m+1} different routes from vertex m to n and F_m different routes to vertex m, so that we have a total of $F_m \cdot F_{n-m+1}$ routes to vertex n that go through vertex m. The only way a route to vertex n can bypass vertex m is if it goes directly to vertex $m+1$ from vertex $m-1$ (by either an H or L move). There are F_{m-1} distinct routes to vertex $m-1$; for each one of these routes, since $n - (m+1) + 1 = n - m$, there will be F_{n-m} different routes from vertex $m+1$ to n for a total of $F_{n-m} \cdot F_{m-1}$ of routes to n bypassing m. Adding up we get the formula to be proved.

2. Show that if m divides n, then F_m divides F_n.
Hint: Apply the formula of Problem 1 with $n = km$.
Solution. Because m divides n there is some integer $k \geq 1$ such that $n = km$. We may assume $k > 1$, otherwise there isn't much to prove. Applying the formula of Problem 1 with $n = km$ we get

$$F_n = F_{km} = F_m \cdot F_{(k-1)m+1} + F_{(k-1)m} \cdot F_{m-1}.$$

(This formula works also if $m = 1$, so $n = k$, by interpreting $F_0 = 0$.) If $k = 2$, the formula tells us that $F_{2m} = F_m \cdot (F_{m+1} + F_{m-1})$, showing that F_m divides F_{2m}. Next, applying it with $k = 3$, using F_m divides F_{2m}, we see that F_m divides F_{3m}. Continuing in this way, we see that F_m divides F_{kn} for $k = 1, 2, 3, 4$, etc.

3. Show that if a positive integer divides two consecutive Fibonacci numbers, it must also divide the Fibonacci number preceding the smaller of the two. In symbols: If d divides F_{n+1} and F_{n+2}, it must also divide F_n. Explain why this implies that if d divides two consecutive Fibonacci numbers, it must be 1. In other words, $gcd(F_n, F_{n+1}) = 1$. Route counting is probably NOT indicated.
Solution. If d divides $F_{n+2} = F_{n+1} + F_n$ and F_{n+1}, it must also divide $F_{n+2} - F_{n+1} = F_n$. But then continuing this process (if $n > 1$), we get d divides F_{n-1} because it divides F_{n+1} and F_n, next it divides F_{n-2}, and so forth until we get all the way to d divides $F_1 = 1$, so $d = 1$.

For the next problems we need some results from number theory; the first one is an easy consequence of the fact that every positive integer > 1 can be written in one, and up to order of the factors, in only one way as a product of primes. The result is: *Suppose a, b, c are positive integers and $gcd(a, c) = 1$. If c divides the product $a \cdot b$, then c divides b.*

The second result we may need is a consequence of the Euclidean algorithm: *Suppose m, n are positive integers. One can always find non negative integers a, b such that either $d = am - bn$ or $d = bn - am$. Except if m is a multiple of n or n of m, both a, b will be positive.*

Can you prove these results? If not, no matter; accept them for now.

4. Show that for any pair F_n, F_m of Fibonacci numbers,

$$gcd(F_n, F_m) = F_{gcd(n,m)}.$$

Hint: Let $D = gcd(F_m, F_n)$ and let $d = gcd(m, n)$. Show that it is enough to prove that D divides F_d. One can assume $d = an - bm$ with a, b positive, or $an = d + bm$.
Solution. Let d be the greatest common divisor of F_n and F_m, $m < n$. We select non-negative integers a, b such that $d = am - bn$ or $d = bn - am$. We may assume it is the former, $d = an - bm$; the other case is identical. The only way b can be 0 is if d is a multiple of n, but then (as the gcd of m, n, $d = n$ and n divides m. By Problem 2, F_n divides F_m, thus $D = F_n = F_d$. So assume $a > 0, b > 0$ and write $an = d + bm$. By the formula of Problem 1 (with n replaced by $d + bm = an$ and m replaced by bm we have

$$F_{an} = F_{d+an} = F_{bm} \cdot F_{d+1} + F_d \cdot F_{bm-1}.$$

Since D divides F_n and F_n divides F_{an}, we see that D divides F_{an}. Similarly, D divides F_{bm}. From the equation above, D divides $F_d \cdot F_{bm-1}$. But $gcd(D, F_{bm-1}) = 1$. In fact, a common divisor of D and F_{bm-1} would also be a common divisor of F_{bm} and F_{bm-1}, thus must be 1. We conclude that D divides F_d. On the other hand, by problem 2, F_d divides F_m and F_n, thus F_d divides D. It follows that $D = F_d$.

5. We want to get the converse of Problem 1; well, almost. Show that if F_m divides F_n and m is not equal to 1 or 2, then m divides n. We may, of course, assume $m \leq n$.
Hint: Problem 4.
Solution. Suppose F_m divides F_n. Then

$$F_m = gcd(F_m, F_n) = F_{gcd(m,n)}.$$

Since $m \neq 1, 2$, we conclude that $m = gcd(m, n)$, thus m divides n.