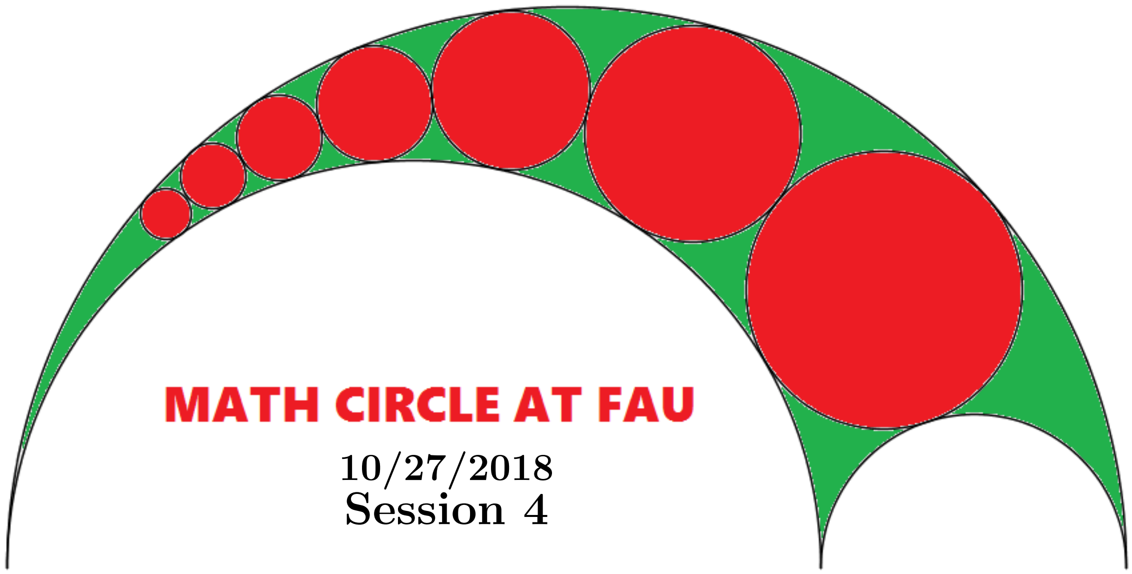


MATH CIRCLE AT FAU

10/27/2018

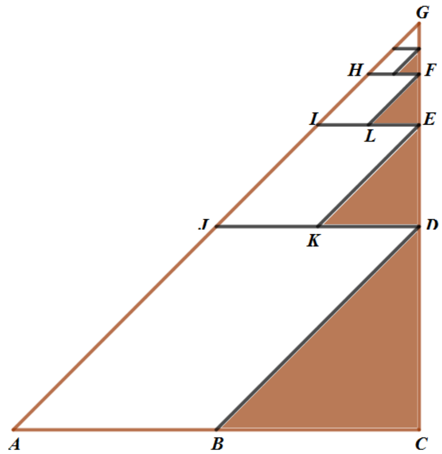
Session 4

RULES

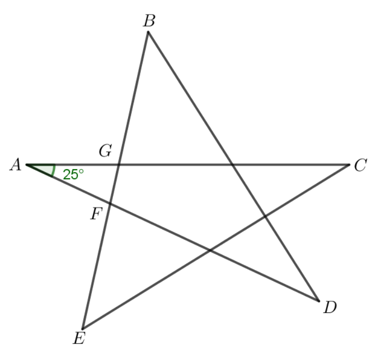
- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.
- Except in case of an emergency, all smart devices are strictly forbidden. No smart phones, smart watches, or time machines. Only smart people.
- Problems might get harder toward the end. **Please don't feel frustrated if you can't do them.** Still, you can ask an organizer for explanations; maybe with a few hints you can work the problem.

1. At the grocery store last week, small boxes of facial tissue were priced at 4 boxes for \$5. This week they are on sale at 5 boxes for \$4. Find the percent decrease in the price per box during the sale. (AJHSME)

2. Points B, D, and J are midpoints of the sides of right triangle ACG. Points K, E, I are midpoints of the sides of triangle JDG, etc. If the dividing and shading process is done 100 times (the first three are shown) and $AC = CG = 6$, then the integer nearest to the total area of the shaded triangles is ? (AMC 8)



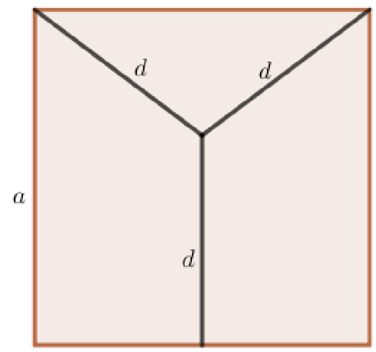
3. In the star shaped figure, the angle at A measures $25°$ and $\angle AFG = \angle AGF$. Find $\angle B + \angle D$. (AMC 8)



4. Show that if the three digit number abc is divisible by 7, then so is $cba - (c - a)$. For example, 133 is divisible by 7, and so is $331 - (3 - 1) = 331 - 2 = 329$. Here is another example: 623 is divisible by 7 and so is $326 - (3 - 6) = 326 - (-3) = 326 + 3 = 329$.

5. A man living in the suburbs and working in the city returns every day by a train that arrives at his suburban station at exactly 5 p.m. His butler picks him up at the station to drive him home; the butler has arranged things so he arrives at exactly 5 p.m. at the station. The route the butler takes to the station is the same as he takes to drive home. One day the man takes an earlier train, arriving at the station at 4. He decides to start walking home, along the route taken always by the butler. Somewhere along the route he meets the butler; he jumps into the car and the butler drives him home arriving 10 minutes earlier than usual. How long had the man be walking before being picked up by the butler? Assume that getting in and out of a car, turning the car around, everything but the driving (and walking) is instantaneous.

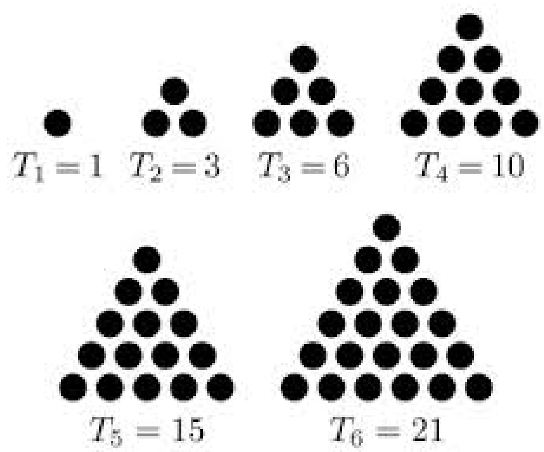
6. A point P inside a square of side length a is at the same distance d from two consecutive vertices and from the side opposite the two vertices. What is a/d?



7. Three friends, Amy, Bertie and Chippy, have some money and not only are they very good friends, but they also are very generous. So Amy decides to give Bertie and Chippy some of her money, after which Bertie and Chippy have twice as much money as before. Then Bertie gives money to Amy and Chippy, doubling their amount of Money. Finally, Chippy gives away money to Amy and Bertie, doubling their amounts. If Chippy started with \$ 72 and at the end has again \$ 72, what is the total amount of money the three friends have? (AMC 8)

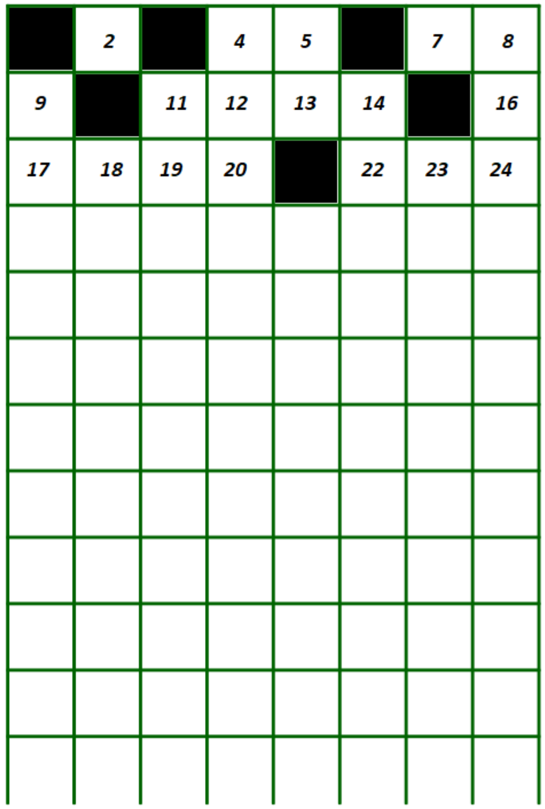
8. Find the area of the trapezoid pictured below where $AB = 23$, $BC = 8$, $CD = 13$ and $DA = 6$.

9. *Triangular numbers* are numbers that can be represented by triangular shapes made out of dots. The first few are



   (a) Show, explain $1 + 2 + \cdots + n = T_n$.
   (b) Find a simple formula for T_n.

10. A rectangular board of 8 columns has squares numbered beginning in the upper left corner and moving left to right so row one is numbered 1 through 8, row two is 9 through 16, and so on. A student shades square 1, then skips one square and shades square 3, skips two squares and shades square 6, skips 3 squares and shades square 10, and continues in this way until there is at least one shaded square in each column. What is the number of the shaded square that first achieves this result? (AJHSME)



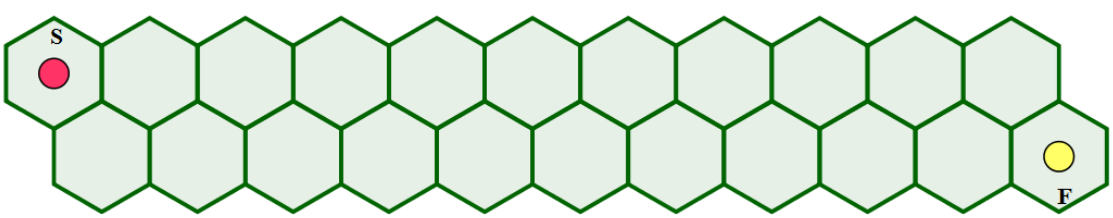
11. This is an almost Fibonacci problem. There used to be a time when people wrote letters to friends and family, instead of texting or email. In those far, far gone days, a person, lets call him Armand, wrote 7 letters to 7 friends. Armand had previously addressed 7 envelopes for the letters written. In how many ways can Armand place **every** single letter into the wrong envelope?

    (As with so many problems one should start with easy cases. For example if there is one letter, one envelope, there are 0 ways of making a mistake. If there are two letters (lets call them L_1 and L_2) and two envelopes (E_1 and E_2) then there is one way; L_1 into E_2 and L_2 into E_1. And so it goes.)

The Fibonacci Section

Some of you have perhaps solved last session's "Buzzy the Bee" problem. As a reminder, here it is, with a solution.

Buzzy the Bee starting at cell S with the red dot at the center wants to get to the cell F, with the yellow dot at the center.



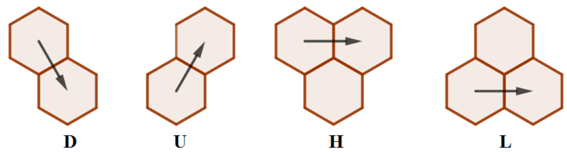
Each step must move to the right and to a neighboring cell. There are four possible types of steps:

D = downward diagonal.

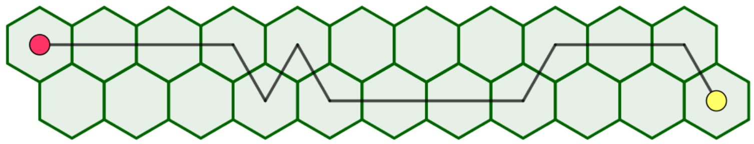
U = upward diagonal.

H = high horizontal step.

L = low horizontal step.



For example, one allowed route is the one shown below:



How many different routes are there from the cell with the red dot to the cell with the yellow dot?

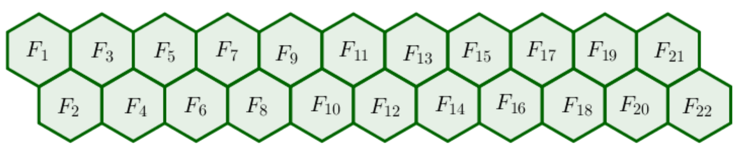
Hint: Starting at the cell with the red dot, enter in each cell the number of different ways it can be reached from the cell with the red dot. It might suggest something.

If you followed the suggestion at the end of the problem, you would discover that the number of different ways to get to the cell n positions to the right of the cell with the red dot was F_n, the n-th Fibonacci number. The Fibonacci numbers are the numbers in the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \dots;$$

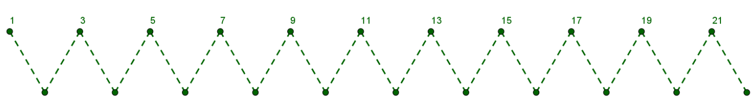
it starts with 1, 1, and from then on every number is the sum of the two preceding ones. The so called recursive definition is $F_1 = 1$, $F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ once $n \geq 3$.

Your honeycomb then looks like

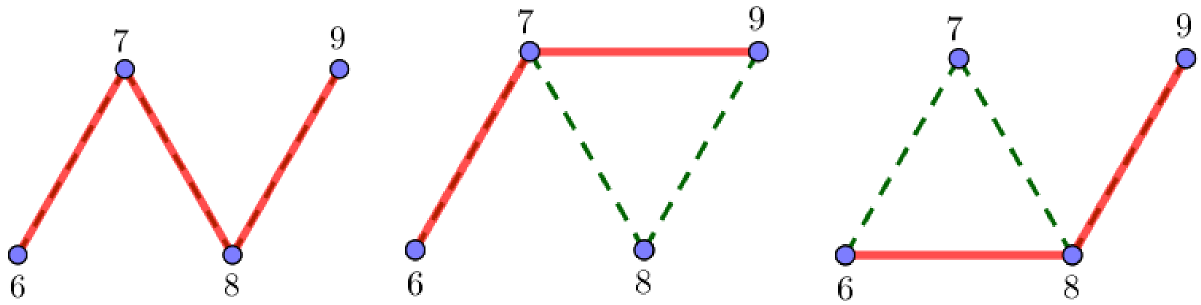


The answer to the question is $F_22 = 17,711$.

The remaining problems build upon this counting of routes. However, the cells are a bit distracting. It's easier perhaps to replace them with dots, and just number the vertices, so the picture looks like



What the argument used to solve the buzzy the bee problem shows is that starting at any vertex, there are exactly F_{k+1} routes to a vertex k positions to the right of it. In other words, if $1 \leq m < n$ then there are F_{n-m+1} different routes from vertex m to vertex n. To give a very simple example the routes from vertex 6 to vertex 9 are:



Because $9-6+1=4$, there are $F_4=3$ different routes.

Here are the problems.

1. By counting routes show that if $1 \le m < n$ then

$$F_n = F_m \cdot F_{n-m+1} + F_{n-m} \cdot F_{m-1}.$$

For example, with $n=21$, $m=12$ so that $n-m=9$, $n-m+1=10$ and $m-1=11$, the formula says

$$F_{21} = F_12 \cdot F_{10} + F_9 \cdot F_{11} = 144 \cdot 55 + 34 \cdot 89 = 10,946.$$

2. Show that if m divides n, then F_m divides F_n.
Hint: Apply the formula of Problem 1 with $n = km$.

3. Show that if a positive integer divides two consecutive Fibonacci numbers, it must also divide the Fibonacci number preceding the smaller of the two. In symbols: If d divides F_{n+1} and F_{n+2}, it must also divide F_n. Explain why this implies that if d divides two consecutive Fibonacci numbers, it must be 1. In other words, $gcd(F_n, F_{n+1}) = 1$. Route counting is probably NOT indicated.

For the next problems we need some results from number theory; the first one is an easy consequence of the fact that every positive integer > 1 can be written in one, and up to order of the factors, in only one way as a product of primes. The result is: *Suppose a, b, c are positive integers and $gcd(a, c) = 1$. If c divides the product $a \cdot b$, then c divides b.*

The second result we may need is a consequence of the Euclidean algorithm: *Suppose m, n are positive integers. One can always find non negative integers a, b such that either $d = am - bn$ or $d = bn - am$. Except if m is a multiple of n or n of m, both a, b will be positive.*

Can you prove these results? If not, no matter; accept them for now.

4. Show that for any pair F_n, F_m of Fibonacci numbers,

$$gcd(F_n, F_m) = F_{gcd(n,m)}.$$

Hint: Let $D = gcd(F_m, F_n)$ and let $d = gcd(m, n)$. Show that it is enough to prove that D divides F_d. One can assume $d = an - bm$ with a, b positive, or $an = d + bm$.

5. We want to get the converse of Problem 1; well, almost. Show that if F_m divides F_n and m is not equal to 1 or 2, then m divides n. We may, of course, assume $m \le n$.
Hint: Problem 4.