MATH CIRCLE, 10/13/2018

LARGE SOLUTIONS

1. Write out row 8 of Pascal's triangle.

   Solution.

   $$1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1.$$

2. Write out all the different ways you can choose three letters from the set $\{a, b, c, d, e, f\}$. When you count the number of ways you did this, does it relate to Pascal's triangle?

   Solution. The different ways are

   $$abc,\ abd,\ abe,\ abf,\ acd,\ ace,\ acf,\ ade,\ adf,\ aef,\ bcd,\ bce,\ bcf,\ bde,\ bdf,\ bef,\ cde,\ cdf,\ cef,\ def,$$

   20 ways in all. If we go to row 6 of Pascal's triangle, and then to the 3rd (element counting from 0) in that row we see that $\binom{6}{3} = 20$.

3. Can you explain why the following is true: The number of ways of choosing 8 objects from a set of 20 equals the number of ways of choosing 8 objects from a set of 19 plus the number of ways of choosing 7 objects from a set of 19. In symbols

   $$\binom{20}{8} = \binom{19}{8} + \binom{19}{7}.$$

   Does this explain how Pascal's triangle works?

   Solution. Say the set of 20 objects is $\{1, 2, 3, \ldots, 20\}$. A set of 8 objects from this set may or may not contain the number 20. If it doesn't contain the number 20, it is really a set of 8 objects from the set $\{1, 2, \ldots, 19\}$. If it contains the number 20, then it is a set of 7 elements from the set $\{1, 2, \ldots, 19\}$ to which we add the number 20. In other words,

   $$\binom{20}{8} = \binom{19}{8} + \binom{19}{7}.$$

   In general if we have n objects and we want to select k, we can select k from among the first $n-1$ elements; this can be done in $\binom{n-1}{k}$ different ways. This gives all subsets not containing the last element. To get all subsets containing the last element, we can select $k-1$ elements from the first $n-1$ elements, which can be done in $\binom{n-1}{k-1}$ ways, and then add to each of these selections the last element. So the general equation is

   $$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1},$$

   which is how you go from row $n-1$ to row n in Pascal's triangle.

4. Notice that each row of Pascal's triangle reads the same from left to right as from right to left. Can you explain this in terms of choosing? That is, for example, can you explain why the number of ways in which you can choose 3 objects from 7 is the same as the number of ways you can choose $4 = 7 - 3$ objects from 7? Why the number of ways you can chose 27 objects from 100 is the same as the number of ways you can choose 73 objects from 100?

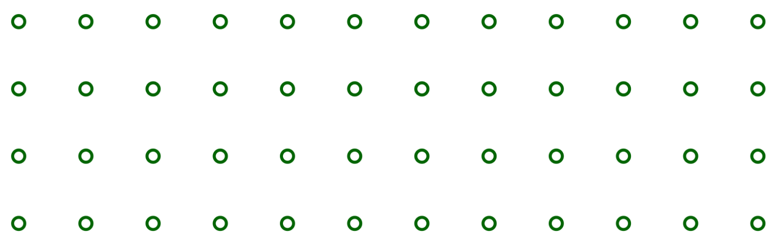
   Solution. One way of choosing a bunch, say k elements, from n is to say which $n-k$ elements you are **not** choosing each time. So here is a way of choosing 3 elements from the set $\{a, b, c, d, e\}$ of 5 elements. Say which elements you are not going to choose: We then have
   Not choosing ab what remains is cde
   Not choosing ac what remains is bde

and so forth.
To select 3 elements is the same as rejecting 2. In general, selecting k elements from n is the same as selecting $n-k$ that will be rejected. We see that $\binom{n}{k} = \binom{n}{n-k}$; each row of Pascal's triangle is symmetric around the middle.

5. Each of the faces of a cube is numbered with a different integer. Each vertex is assigned a "vertex number," which is the sum of the numbers of the faces which intersect in that vertex. Finally all the vertex numbers are added. What is the largest number that must always divide the sum of the vertex numbers, no matter how the faces were numbered? For example, if there is some way in which you can get the final sum to be a prime number, then the answer should be 1. In case you want to experiment, the last page of this booklet allows you to cut out a cube. Glue is available.

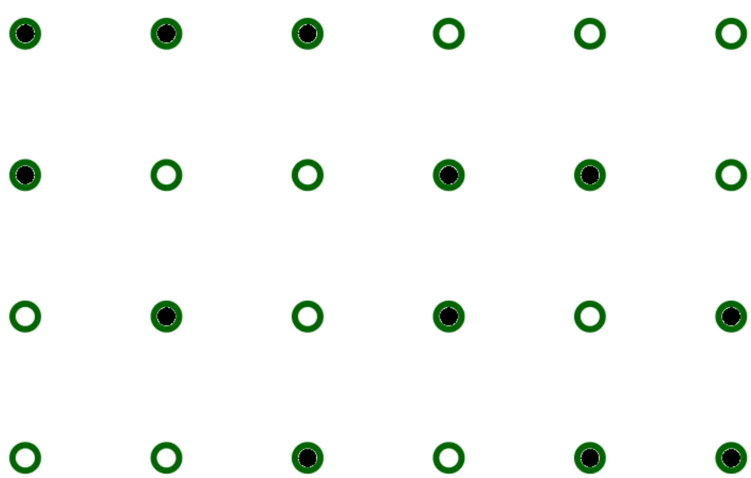
   Solution. The cube has 6 sides, so let us denote the numbers by $a_1, a_2, a_3, a_4, a_5, a_6$. Since face 1 has 4 vertices, the number a_1 is part of the vertex number of the four vertices of face 1. Similarly a_2 is part of the vertex number of each of the four vertices of face 2. And the same for the other twelve numbers. That means when we add up all the vertex numbers, each one of a_1 to a_6 gets added four times so that the final sum is $4(a_1 + a_2 + \cdots + a_6)$. Now $a_1 + \cdots a_6$ could be any number ≥ 6, so the answer is that the largest number that must always divide the sum of the vertex numbers is $\boxed{4.}$

6. Dots are arranged in a rectangular grid having 4 rows and n columns. The picture below shows such a grid with $n = 12$

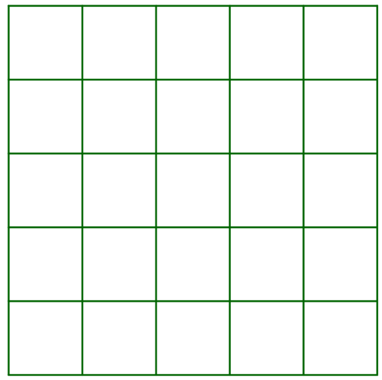


   You are to color some of the dots black, leave others as they are so that no four black dots, or no four white dots, form a rectangle (or square) with horizontal and vertical sides. What is the largest value of n (number of columns) for which this task can be accomplished? What is the answer if instead of 4 rows of dots we have 5 rows? 20 rows?

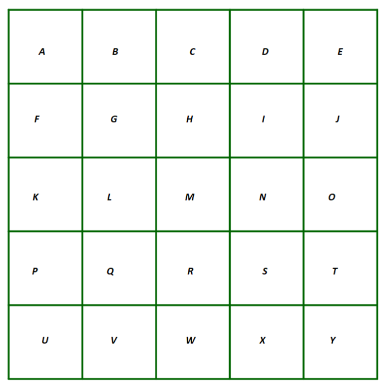
   Solution. The answer for 4 rows is $n = 6$. For each column we have $\binom{4}{2} = 6$ different ways of coloring the dots; so that after coloring the 6th column we must have a repetition. The picture below shows a solution for $n = 6$. In general, for k rows the answer is $\binom{k}{2}$.



7. The picture below shows a 5×5 square partitioned into twenty five 1×1 squares. Find the total number of squares of any size in the picture.



Solution. Every square can be identified by the top-rightmost of its 1×1 sub squares and by its size. So if we identify each 1×1 square by a letter, say as follows:



then every square of a given size is identified by the letter corresponding to its 1×1 sub-square in the rightmost top position. There is one square of size 25; the one identified by A. There are 4 squares of size 16; with top rightmost squares with the letters A, B, F and G. Squares of size 9 can be identified by the letters A, B, C, F, G, H, K, L and M; there are 9 of them. Squares of size 4 are identified by the letters $A, B, C, D, F, G, H, I, K, L, M, N, P, Q, R, S$; there are 16 such squares. Finally, there are 25 1×1 squares. The total number of squares is thus

$$1 + 4 + 9 + 16 + 25 = \boxed{55.}$$

8. Freddy, the so-so chess player, played against Neddy and won 45% of all the games played. Neddy was about to leave, but Freddy convinced Neddy to play 8 more games, of which Freddy won 6. At this point Freddy had won half of the games he played with Neddy. How many games did Freddy play with Neddy? (AMC 8–Sort of).

Solution. Let x be the number of games played. Then Freddy won 45% of all but the last 8 games; that is $0.45(x-8) = 0.45x - 3.6$. If we add the last 6 games won, that means that Freddy won $0.45x + 2.4$ games. Since that is half of all the games, we get $0.45x + 2.4 = 0.5x$, from which we can solve to get $x = 48$.

9. Ms. Nakamura's class has 25 students, 10 boys and 15 girls. Of the students, 4 boys and 7 girls are excellent singers; the rest of the students are just so-so. Ms. Nakamura has to assemble a cast for a production of an opera. She needs 2 boys and 3 girls with excellent voices for the lead roles, and then a chorus of 5 boys and 5 girls from among the remaining students, making sure that the excellent singers not chosen for the lead roles are part of the chorus. In how many different ways can such a cast be assembled?

Solution. Ms. Nakamura has to choose 2 boys from 4, 3 girls from 7, for the lead roles. For the chorus, since the two remaining boys and the 4 remaining girls are automatically in it, she has to select 3 boys from

6 and 2 girls from 11. The total number of different ways this can be done is

$$\binom{4}{2} \times \binom{7}{3} \times \binom{6}{3} \times cbn112 = 6 \times 35 \times 20 \times 55 = 231000 \text{ different ways.}$$

10. Find the sum of all three digit numbers all of whose digits are odd.

Solution. Let us place all of these numbers in a column. There are $5 \times 5 \times 5 = 125$ such numbers. So things could look like

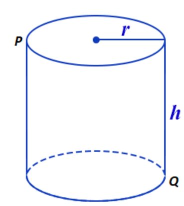
$$\begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 3 \\ . & . & . \\ 9 & 9 & 9 \end{array}$$

In this three columns array of numbers, each digit appears 25 times in each column. The array can be written in a more precise way as

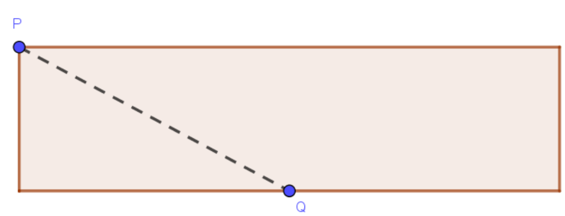
$$\begin{array}{ccccc} 100 & + & 10 & + & 1 \\ 100 & + & 10 & + & 3 \\ \cdots & & & & \\ 900 & + & 90 & + & 9 \end{array}$$

The sum of the numbers in the last column, the digits column will be $25(1+3+5+7+9) = 25 \cdot 25 = 625$; in the second column , the tens column, $25(10+30+50+70+90) = 25 \cdot 250 = 6250$; in the first column, $25(100+300+500+700+900) = 25 \cdot 2500 = 62500$. The total sum is thus $625 + 6250 + 62500 = 69375$.

11. The picture shows a cylinder of height $h = 4$ cm and circumference 6 cm. The point P on the top rim is diametrically opposite from point Q on the bottom rim. What is the shortest distance from P to Q along the surface of the cylinder?



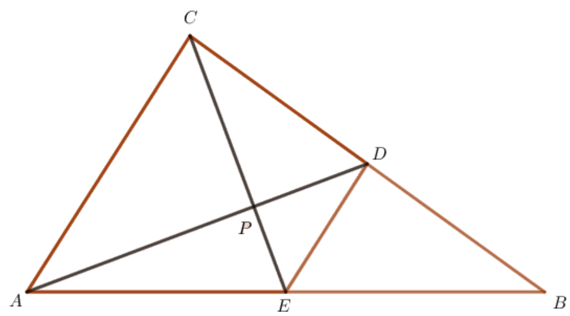
Solution. If we cut the cylinder along a line from P going vertically downward (parallel to the axis) we can flatten out the cylinder into a 6×4 rectangle:



The shortest distance from P to Q is given by the dotted line. It can be computed by the Theorem of Pythagoras. It works out to

$$d = \sqrt{3^2 + 4^2} = 5.$$

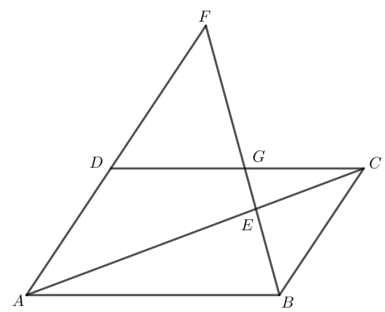
12. In triangle $\triangle ABC$, medians $\overline{AD}$ and $\overline{CE}$ intersect at P. If $PE = 1.5$, $PD = 2$ and $DE = 2.5$, what is the area of the quadrilateral $AEDC$? (AMC 10A)



What do you know about the medians of a triangle? Ask one of the organizers for a hint, perhaps.

Solution. The hint that makes this easy is the medians intersect in a ratio of $1:2$ so that one has $AP = 4$ and $CD = 3$. Moreover the triangles $\triangle APC$ and $\triangle DPE$ are similar so $AC = 2 \cdot DE = 5$. It now helps to recall that a $3, 4, 5$ triangle is a right triangle, so that $\triangle APC$, $\triangle APF$, $\triangle DPE$, and $\triangle CPD$ are right triangles of areas 6, 3, 1.5, and 3, respectively. Adding these areas up we get the area of the quadrilateral as $6 + 3 + 1.5 + 3 = \boxed{13.5}$

13. A line drawn from vertex B of parallelogram $ABCD$ intersects the extension of side AD at F, side CD at G, and diagonal AC at E. If $EF = 32$ and $GF = 24$, what is the length of BE?



14. Find the smallest natural number n with the following properties: **(a)** Its last digit is 6. **(b)** If we remove its last digit and place it in front of the number, we get a number 4 times as large as the original number.

Solution. The easiest solution is multiplying. If the number is say $abc6$ then we know that

$$\begin{array}{ccccc} & a & b & c & 6 \\ \times & & & & 4 \\ \hline & 6 & a & b & c \end{array}$$

We see c must be 4. So we keep the number of digits open and start the multiplication as

$$\begin{array}{ccccc} & - & - & 4 & 6 \\ \times & & & & 4 \\ \hline & - & - & - & 4 \end{array}.$$

Since the last number of the product is 4, the number 4 goes before the 6 in the original number. If we now multiply again by 4 we can say $4 \times 6 = 24$, so 4 carry 2, $4 \times 4 + 2 = 18$, the next number is 8. Our picture looks now like

$$\begin{array}{ccccc} & - & 8 & 4 & 6 \\ \times & & & & 4 \\ \hline & - & - & 8 & 4 \end{array}$$

We were carrying 1 from $4 \times 4 + 2 = 18$, so the next product gives us $4 \times 8 + 1 = 33$ so the number we are looking for is now $\ldots 3846$ and we are carrying 3. Once again $4 \times 3 + 3 = 15$, so we now have the top row of the product looking like 53846, carry 1. Next $4\times 5+1=21$, so we now have for the number $\dots 153486$, carry 2. Next $4\times 1+2=6$,and having arrived at a 6 in front, we are done. The product looks like

$$\begin{array}{rcccccc} & 1 & 5 & 3 & 4 & 8 & 6 \\ \times & & & & & & 4 \\ \hline & 6 & 1 & 5 & 3 & 4 & 8 \end{array}$$

For an alternative, perhaps more sophisticated solution, consider that in decimal notation we can write $n=10^k a_k+\cdots+10a_1+6$. Getting rid of the last 6 and placing it in front produces $6\cdot 10^k+10^{k-1}a_k+\cdots+a_1$. We want

$$6\cdot 10^k+10^{k-1}a_k+\cdots+a_1=4\left(10^k a_k+\cdots+10a_1+6\right)=4\left(10^k a_k+\cdots+10a_1\right)+24.$$

Rearranging,

$$6\cdot 10^k-24=4\left(10^k a_k+\cdots+10a_1\right)-\left(10^{k-1}a_k+\cdots+a_1\right)=39\left(10^{k-1}a_k+\cdots+a_1\right)=3\cdot 13\left(10^{k-1}a_k+\cdots+a_1\right).$$

Dividing by 3.

$$2\cdot 10^k-8=13\left(10^{k-1}a_k+\cdots+a_1\right).$$

It now basically reduces to finding k $2\cdot 10^k-8$ is divisible by 13. A bit of trial and error does it. One can notice that $2\cdot 10^k-8=199\dots 92$ and divide this expression by 13, or (easier I think) one can use modular arithmetic to find the first k if any such that

$$2\cdot 10^k\equiv 8 \pmod{13}, \quad \text{equivalently, } 10^k\equiv 4 \pmod{13}.$$

We get

$$\begin{aligned} 10^2 &\equiv 9 \pmod{13} \\ 10^3 &\equiv 1 \pmod{2}13 \\ 10^4 &\equiv 3 \pmod{13} \\ 10^5 &\equiv 4 \pmod{13} \end{aligned}$$

So $k=5$ is what we were looking for. So now

$$10^{k-1}a_k+\cdots+a_1=\left(2\cdot 10^5-8\right)/13=15384$$

so the number in question is $\boxed{153{,}846.}$

15. Find ALL three digit integers x with the property that x is divisible by 11 and $x/11$ is equal to the sum of the squares of the digits of x. Are there even such numbers?

Solution. Suppose $n=abc=100a+10b+c$. It will be divisible by 11 if and only if one of the two conditions hold:

(a) $a+c<11$ and $b=a+c$. In this case $x/11=10a+c$.

(b) $a+c\geq 11$ and $b=a+c-11$. In this case $x/11=10(a-1)+c$.

Let us find all numbers of the first form that could satisfy the condition. So we assume $a+c<11$ and we want to find digits a,c so that

$$a^2+(a+c)^2+c^2=10a+c.$$

We can expand this and write it as a quadratic equation for a in the form

$$2a^2+2(c-5)a+2c^2-c=0. \tag{1}$$

The first thing to observe is that all terms but possibly the last term are even, so the last term must also be even. In other words, c must be even. We may also notice that if we apply the quadratic formula, the discriminant (the thing inside the square root), will be

$$4(c-5)^2-8(2c^2-c)=100-12c^2-32c.$$

For real solutions (not to mention integer solutions) we must have a non-negative discriminant, thus $12c^2 + 32c \leq 100$. The only digits satisfying this are 0 and 1 and, as we saw, 1 is out (being odd). Plugging $c = 0$ into the equation (1) for a we get $2a^2 - 10a = 0$ or $a = 5$. (We also get $a = 0$, but 000 does not count as a three digits integer). So there is only one solution of type a, namely 550.
Assume now $a + c \geq 11$ and we want to find a, c so that

$$a^2 + (a + c - 11)^2 + c^2 = 10(a - 1) + c. \tag{2}$$

This equation can be expanded to:

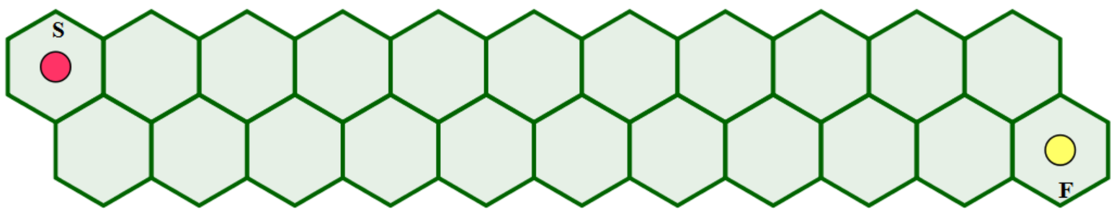
$$2a^2 + 2(c - 16)a + 2c^2 - 23c + 131 = 0.$$

Parity considerations this time force c to be odd. Looking again at the discriminant, one sees that one must have $c \leq u$. Since c must be odd, this leaves $c = 1$ or $c = 3$. With $c = 1$ we cannot possible have $a + c \geq 11$ (and the equation for a does not have integer roots anyway). With $c = 3$, (2) has the solutions $a = 5, 8$. Now $5 + 3 < 11$, so the solution that works (and the only one) is $a = 5, c = 3$, thus $b = 8 + 5 - 11 = 0$.

The only two numbers satisfying the condition are 550 and 803.

16. The trips of Buzzy the Bee.

Buzzy the Bee starting at cell S with the red dot at the center wants to get to the cell F, with the yellow dot at the center.



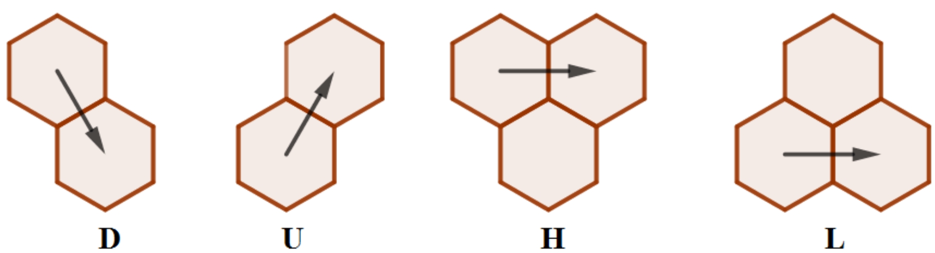
Each step must move to the right and to a neighboring cell. There are four possible types of steps:

D = downward diagonal.

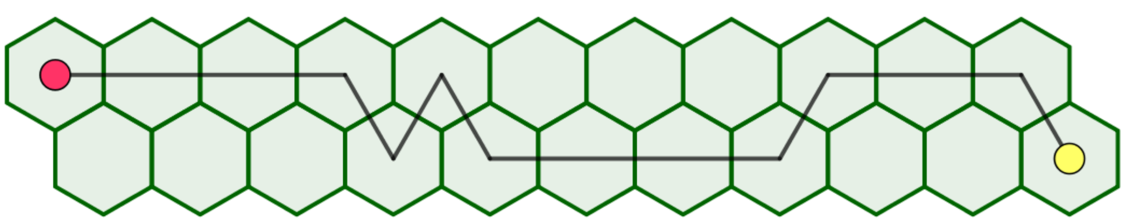
U = upward diagonal.

H = high horizontal step.

L = low horizontal step.

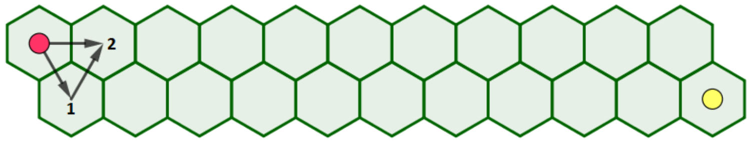


For example, one allowed route is the one shown below:



How many different routes are there from the cell with the red dot to the cell with the yellow dot?

Hint: Starting at the cell with the red dot, enter in each cell the number of different ways it can be reached from the cell with the red dot. It might suggest something. Here is how you would start:



Justify your solution (it is not enough to guess).

Solution. I have to write this up in detail. If you number the cells from left to right starting with a 1 in the cell with the red dot, it is fairly clear that the number of to get to cell $n \geq 2$ you either have to pass through cell $n-1$ or cell $n-2$ just before you can get to cell n. Thus the number of different roots to reach cell n works out to the n-th Fibonacci number F_n. The answer to the question is $F_2 2 = 17,711$.