MATH CIRCLE AT FAU

10/13/2018
Session # 3

RULES

- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.
- Except in case of an emergency, all smart devices are strictly forbidden. No smart phones, smart watches, or time machines. Only smart people.
- Problems get harder toward the end, especially after problem 10. **Please don't feel frustrated if you can't do them. Some may require 10th, 11th or 12th grade mathematics.** Still, you can ask an organizer for explanations; maybe with a few hints you can work the problem.

Fun thing to know, which may play a part in some problems.

Some of you may know this already. Otherwise, read on!

In how many ways can we choose 3 objects from a set of 5? 4 objects from a set of 7? Is there a formula? The answer is in the famous *Pascal's Triangle*, a triangle of numbers in which each row starts with 1, ends with 1, and every number in between is the sum of the two numbers to its left and right above it: Here it is up to row 7:

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

We number rows and entries in each row starting with 0. So row 0 is the top row with only the number 1 in it; row 1 has two 1's; row 2 consists of 1, 2, 1; and so forth. Pascal's triangle plays a role in the choosing problem. To illustrate, I will answer the very first question made above; in how many ways can we choose 3 objects from a set of 5? To be more specific, in how many ways can we choose three letter from the set $\{a, b, c, d, e\}$? (Order does not matter) Here are the ways:

$$abc, \quad abd, \quad abe, \quad acd, \quad ace, \quad ade, \quad bcd, \quad bce, \quad bde, \quad cde$$

10 ways in all. Now go to row 5 of Pascal's triangle and go to position 3 of that row (remembering to count from 0). You find the number 10 there. Coincidence? Not really.

The entries in the triangle are sometimes known as *combinatorial numbers*. There is a symbol for the k-th entry in row n; it is $\left(\begin{array}{c} n \\ k \end{array}\right)$. For example

$$\left(\begin{array}{c} 5 \\ 2 \end{array}\right) = 10, \quad \left(\begin{array}{c} 7 \\ 3 \end{array}\right) = 35, \quad \left(\begin{array}{c} 5 \\ 5 \end{array}\right) = 1, \quad \left(\begin{array}{c} 6 \\ 0 \end{array}\right) = 1, \quad \left(\begin{array}{c} 4 \\ 5 \end{array}\right) \text{ there is no such thing.}$$

The number $\left(\begin{array}{c} n \\ k \end{array}\right)$ tells you, in fact, in how many different ways you can choose k objects from a set of n.

1 THE PROBLEMS.

1. Write out row 8 of Pascal's triangle.

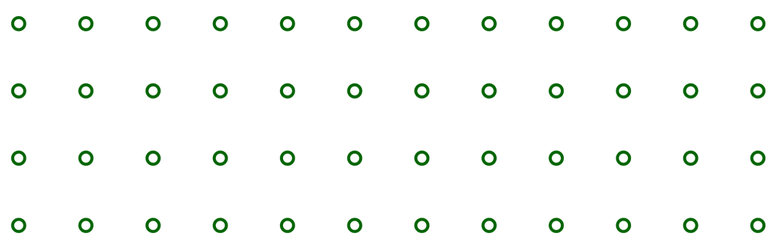
2. Write out all the different ways you can choose three letters from the set $\{a, b, c, d, e, f\}$. When you count the number of ways you did this, does it relate to Pascal's triangle?

3. Can you explain why the following is true: The number of ways of choosing 8 objects from a set of 20 equals the number of ways of choosing 8 objects from a set of 19 plus the number of ways of choosing 7 objects from a set of 19. In symbols
$$\binom{20}{8} = \binom{19}{8} + \binom{19}{7}.$$
Does this explain how Pascal's triangle works?

4. Notice that each row of Pascal's triangle reads the same from left to right as from right to left. Can you explain this in terms of choosing? That is, for example, can you explain why the number of ways in which you can choose 3 objects from 7 is the same as the number of ways you can choose $4 = 7 - 3$ objects from 7? Why the number of ways you can chose 27 objects from 100 is the same as the number of ways you can choose 73 objects from 100?

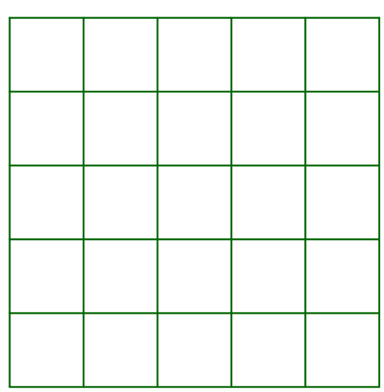
5. Each of the faces of a cube is numbered with a different integer. Each vertex is assigned a "vertex number," which is the sum of the numbers of the faces which intersect in that vertex. Finally all the vertex numbers are added. What is the largest number that must always divide the sum of the vertex numbers, no matter how the faces were numbered? For example, if there is some way in which you can get the final sum to be a prime number, then the answer should be 1. In case you want to experiment, the last page of this booklet allows you to cut out a cube. Glue is available.

6. Dots are arranged in a rectangular grid having 4 rows and n columns. The picture below shows such a grid with $n = 12$



You are to color some of the dots black, leave others as they are so that no four black dots, or no four white dots, form a rectangle (or square) with horizontal and vertical sides. What is the largest value of n (number of columns) for which this task can be accomplished? What is the answer if instead of 4 rows of dots we have 5 rows? 20 rows?

7. The picture below shows a 5×5 square partitioned into twenty five 1×1 squares. Find the total number of squares of any size in the picture.

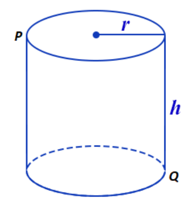


8. Freddy, the so-so chess player, played against Neddy and won 45% of all the games played. Neddy was about to leave, but Freddy convinced Neddy to play 8 more games, of which Freddy won 6. At this point Freddy had won half of the games he played with Neddy. How many games did Freddy play with Neddy? (AMC 8–Sort of).

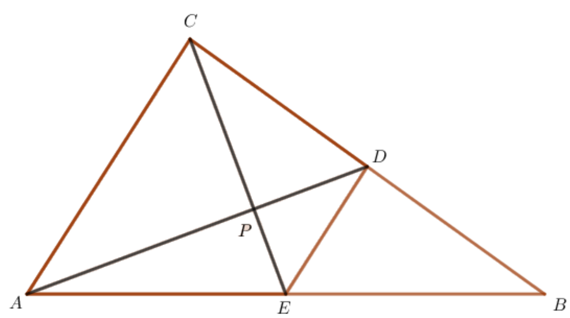
9. Ms. Nakamura's class has 25 students, 10 boys and 15 girls. Of the students, 4 boys and 7 girls are excellent singers; the rest of the students are just so-so. Ms. Nakamura has to assemble a cast for a production of an opera. She needs 2 boys and 3 girls with excellent voices for the lead roles, and then a chorus of 5 boys and 5 girls from among the remaining students, making sure that the excellent singers not chosen for the lead roles are part of the chorus. In how many different ways can such a cast be assembled?

10. Find the sum of all three digit numbers all of whose digits are odd.

11. The picture shows a cylinder of height $h = 4$ cm and circumference 6 cm. The point P on the top rim is diametrically opposite from point Q on the bottom rim. What is the shortest distance from P to Q along the surface of the cylinder?

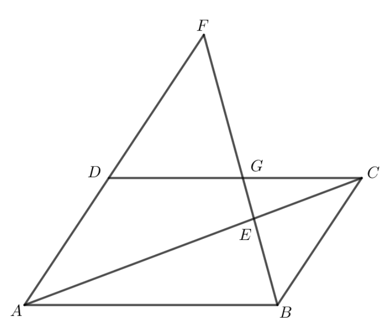


12. In triangle $\triangle ABC$, medians $\overline{AD}$ and $\overline{CE}$ intersect at P. If $PE = 1.5$, $PD = 2$ and $DE = 2.5$, what is the area of the quadrilateral $AEDC$? (AMC 10A)



What do you know about the medians of a triangle? Ask one of the organizers for a hint, perhaps.

13. A line drawn from vertex B of parallelogram $ABCD$ intersects the extension of side AD at F, side CD at G, and diagonal AC at E. If $EF = 32$ and $GF = 24$, what is the length of BE?

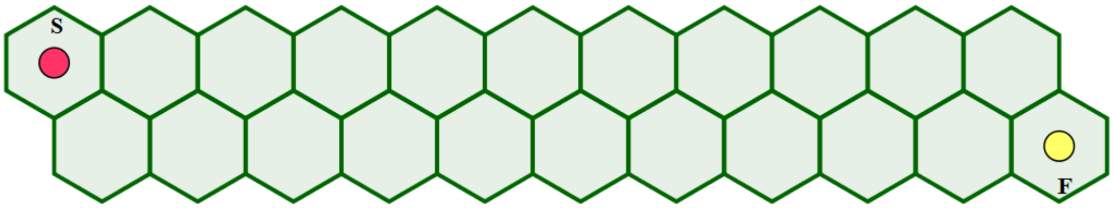


14. Find the smallest natural number n with the following properties: **(a)** Its last digit is 6. **(b)** If we remove its last digit and place it in front of the number, we get a number 4 times as large as the original number.

15. Find ALL three digit integers x with the property that x is divisible by 11 and $x/11$ is equal to the sum of the squares of the digits of x. Are there even such numbers?

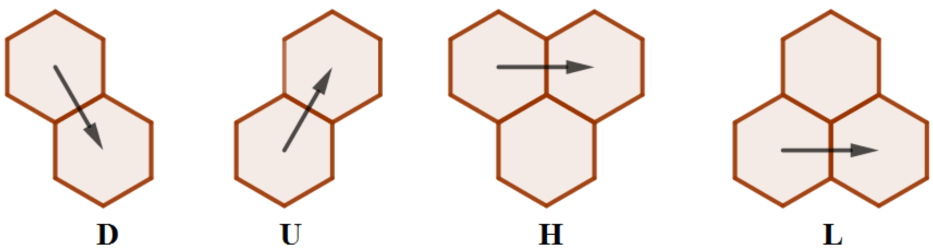
16. **The trips of Buzzy the Bee.**

Buzzy the Bee starting at cell S with the red dot at the center wants to get to the cell F, with the yellow dot at the center.

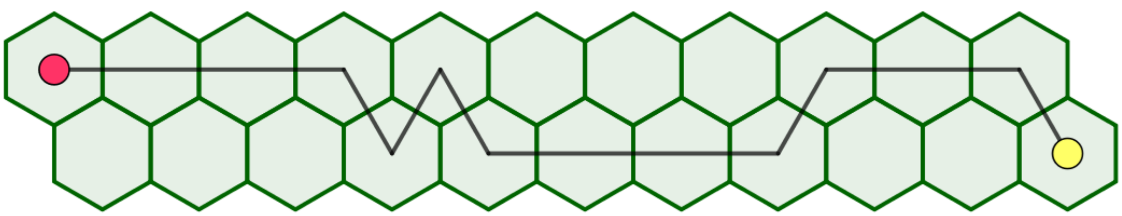


Each step must move to the right and to a neighboring cell. There are four possible types of steps:

D = downward diagonal.
U = upward diagonal.
H = high horizontal step.
L = low horizontal step.

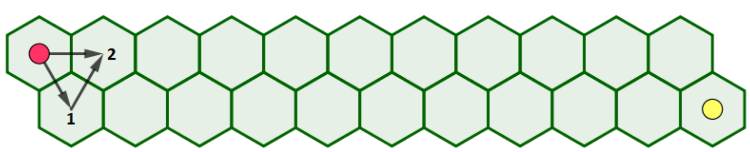


For example, one allowed route is the one shown below:



How many different routes are there from the cell with the red dot to the cell with the yellow dot?

Hint: Starting at the cell with the red dot, enter in each cell the number of different ways it can be reached from the cell with the red dot. It might suggest something. Here is how you would start:



Justify your solution (it is not enough to guess).