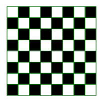
MATH CIRCLE Session # 2, 9/29/2018

SOLUTIONS

1. **The n-Queens Problem.**

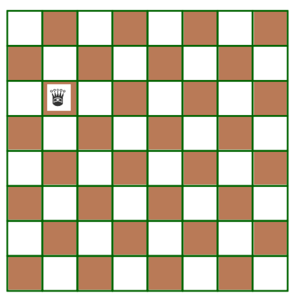
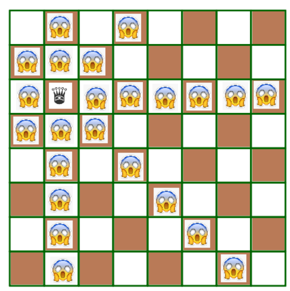
You do NOT need to know how to play chess to work this problem!

This is a classical problem; to look it up today on the internet would be cheating. Here is how it goes. A standard chess board is a square board divided into 64 squares, in 8 rows of 8 squares, in alternating colors (black and white, for example): 

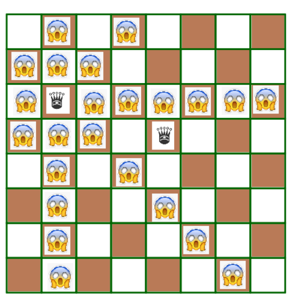
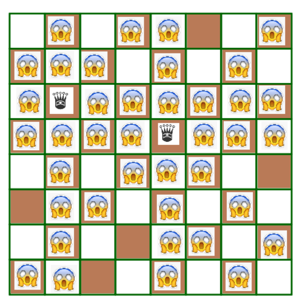
All you need to know about the game of chess is that one of the pieces, a very powerful one, is the queen . The queen can move on the board either diagonally or across rows or columns; any opponent chess piece in the path of the queen could be "eaten" (eliminated). The *eight queen problem* was first posed some hundred fifty years ago, here it is:

Place 8 queens on the chessboard so that no queen threatens another queen; that is no two queens should be in the same row, column or diagonal.

As an example, if I place a queen as in the following picture on the left, the picture on the right has an emoji in all the threatened squares:

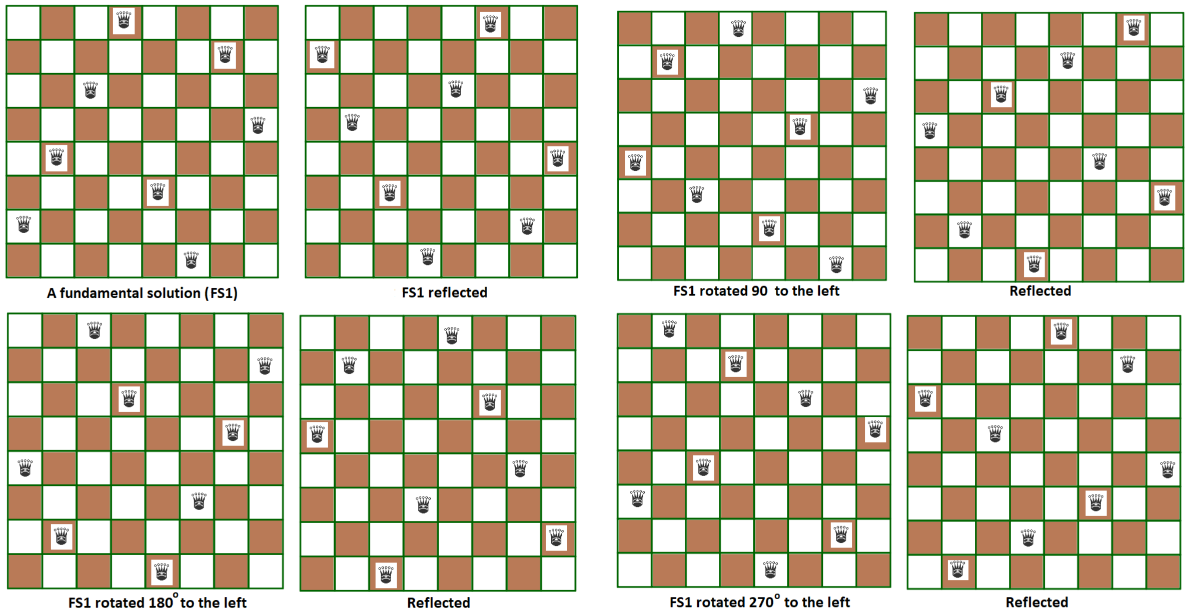
 

If I now place another queen in one of the free squares, more squares are threatened. For example, if I place a queen as in the picture on the left below, the threatened squares are shown on the right.

It has been determined (proved) that there are 12 so called fundamental solutions of the eight queens problem, and all others (to a total of 92!) can be obtained from one of these twelve by either rotating the board or reflecting. The next page shows one of the fundamental solutions, and 8 others you can get from it by rotating and reflecting. **Can you find other solutions?** Maybe you should start with smaller boards. It is obviously NOT possible to place two queens on a 2×2 board so they don't threaten each other. It is also not hard to see that it is it is NOT possible to place three queens on a 3×3 board without one of them threatening another one. What about placing four queens on a 4×4 board? Five on a 5×5 board. There are solutions for placing n queens on an $n \times n$ board once n is greater than 3. But as n increases, one knows less and less about how many solutions there are. For boards larger than 27×27 not much is known.

Here are 8 solutions of the 8 queens problem. The first one is fundamental. The others are obtained either by rotation (as indicated), or by reflection across the horizontal middle line of the board to its left.



(a) Can you find other solutions?

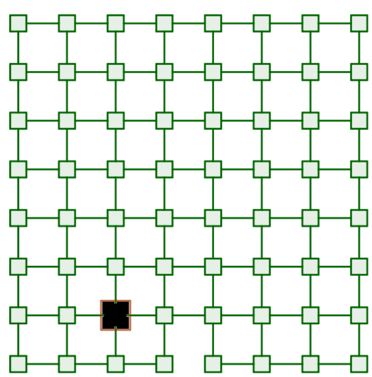
(b) How **many** solutions are there for the 4×4, the 5×5, the 6×6, and the 7×7 puzzle?

(c) Can one place **9** queens on the 8×8 board so none threatens any other? WHY?

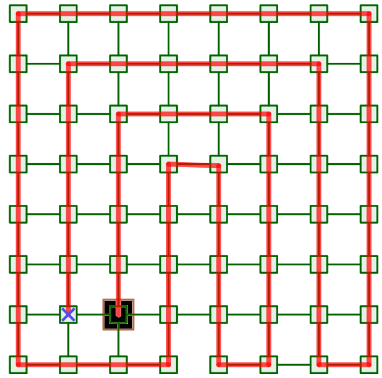
(d) Can you find solutions when $n \geq 9$?

Solution. Not really a solution. For more information, google 8 queens puzzle.

2. This is another old puzzle. The picture below is a map with each little square representing a town; 64 towns in all. The lines connecting them are roads. **Notice there is no direct road connecting the fourth town of the bottom row with the fifth town of the same road.** The puzzle consists in starting from the large black town and visit all other towns once and only once in **fifteen** straight trips.



Solution.



3. Connie multiplies a number by 2 and gets 60 as her answer. However, she should have divided the number by 2 to get the correct answer. What is the correct answer? (AMC 8)

   Solution. The original number musthave been $60/2 = 30$, thus the answer should have been $15 = 30/2$.

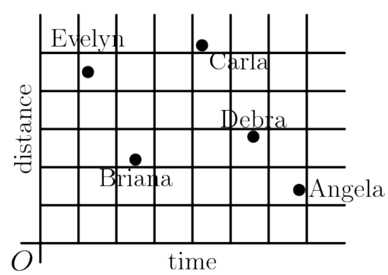
4. Bill walks $\frac{1}{2}$ mile south, then $\frac{3}{4}$ mile east, and finally $\frac{1}{2}$ mile south. How many miles is he, in a direct line, from his starting point? Your answer (in miles) should be in the from $\dfrac{a}{b}$, where a, b are integers with no common divisors other than 1.(AMC 8)

   Solution. Drawing a picture is a good idea. If we set up cartesian coordinates with Bill starting at $(0,0)$, heends at $(-1, 3/4)$. The distance is thus
   $$\sqrt{1 + \frac{9}{16}} = \frac{5}{4}.$$

5. How many different isosceles triangles have integer side lengths and perimeter 23? (AMC 8)

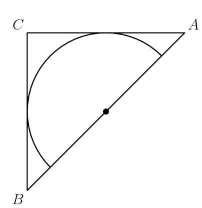
   Solution. If the sides of the triangle are a, a, b then a, b must satisfy $2a > b$ and $2a + b = 23$. All choices of a, b satisfying these conditions work. Now $2a > b, 2a + b = 23$ implies $2a \geq \lceil 23/2 \rceil$, so $a \geq 6$. We also must have $a \leq 11$. But there are no other restrictions on a so that we have a total of 6 such triangles.

6. The results of a cross-country team's training run are graphed below. Which student has the greatest average speed? (AMC 8)

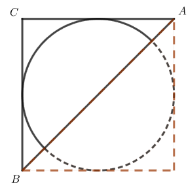


   Solution. Since speed is distance/time, so the slope from the origin to the student, the answer is that Evelyn has the greatest average speed.

7. Isosceles right triangle ABC encloses a semicircle of area 2π. The circle has its center O on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle ABC? (AMC 8)

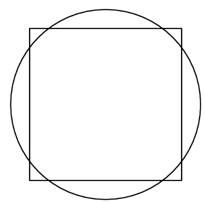


Solution. We can double the isosceles triangle to get a square, the semicircle to get a circle:



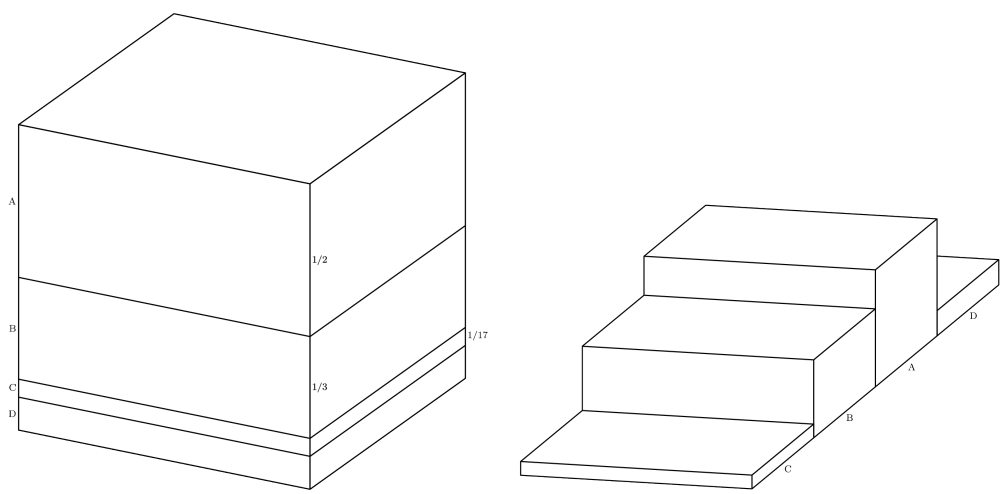
From this we see at once that the length of the leg of the triangle is twice that of the radius of the semicircle. The area of the semicircle being 2π implies its radius is 2 so the area of the isosceles triangle is $\frac{1}{2}(4 \times 4) = 8$.

8. A square with side length 2 and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle? (AMC 8)



Solution. If the area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square, then square and circle have the same are. The area of the square being $2 \times 2 = 4$, this means that with r the radius of the circle $\pi r^2 = 4$, thus $r = 2/\sqrt{\pi}$.

9. A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube. The first cube is $\frac{1}{2}$ foot from the top face. The second cut is $\frac{1}{3}$ foot below the first cut, and the third cut is $\frac{1}{17}$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the second diagram. What is the total surface area of this solid in square feet? (AMC 8)



Solution. The pictures, copied from the AMC 8 sheet, are terribly out of scale, which makes this problem harder than it seems. The left and right sides have the same area as the left and right sides of the original cube, which is 1 square foot each, giving a total of 2 sq.ft. The bottom is made up out of four sides of the cube, has an area of 4 sq.ft. The same can be said of the top faces, thes are made up out of four faces of the cube, so provide an additional 4 sq.ft. Finally the non lateral vertical portions have an area of

$$\left\{\frac{1}{17} + (\frac{1}{3} - \frac{1}{17}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{2} - [1 - (\frac{1}{17} + \frac{1}{2} + \frac{1}{3})]) + [1 - (\frac{1}{17} + \frac{1}{2} + \frac{1}{3})]\right\} \times 1 = \frac{1}{2} + \frac{1}{2} = 1.$$

Adding up, we see that area is $2+4+4+1=11$.

10. In the computation below, A, B, C, D, E, F, G, H, K, L are distinct (so no two are equal) digits and AB is double a prime number. Find A,B,..., H,K,L.

$$\begin{array}{cccccc} & \mathrm{A} & \mathrm{B} & \mathrm{C} & \mathrm{D} & \mathrm{E} \\ + & \mathrm{E} & \mathrm{D} & \mathrm{C} & \mathrm{B} & \mathrm{A} \\ \hline & \mathrm{F} & \mathrm{G} & \mathrm{H} & \mathrm{K} & \mathrm{L} \end{array}$$

Solution. The possible values of AB are

$$10, 14, 22, 26, 34, 38, 46, 58, 62, 74, 82, 86, 94.$$

Of these we can eliminate 10 at once; otherwise $E = L$. One can see that the following relations must hold:

$$\begin{aligned} 0 < A+E &= L \le 9, \\ B+D &= K \quad \text{or} \quad B+D = 1K, \\ B+D &= G \quad \text{or} \quad B+D = 1G \quad \text{or} \quad B+D+1 = G \quad \text{or} \quad 1G, \end{aligned}$$

Since $G \neq K$, it has to be $B+D+1 = 1G$, implying that

$$C+C = 1H \quad \text{or} \quad C+C+1 = 1H.$$

Since there cannot be a "carry" in the sum has only five digits, the sum of the first digits of the summands cannot be ≥ 10. since that sum is the same as the sum of the two last digits, namely $A+E$, we see that $B+D = K$ or $B+D = 1K$ with K at,most 8. If the former then the sum of the leading digits must be F. However, that's again $A+E$ so it would imply $F = K$. Thus $B+D = 1K$. Looking at the appearance of B, D as second digits of the summands, we conclude that to avoid $G = L$ we need to have $C+C+1 \geq 10$, so that $C+C \geq 10$ (since even) and $C+C+1 = 1H$, and H is odd.

One of the digits must equal 0. We try next to determine which. The following digits are out:

A, E since $A+E = L \neq A, E$.

B,D since $B+D = K$ or $1K \neq B, D$.

C since $C+C \geq 10$.

F,L since $L = A+E$, $F = A+E+1$.

This leaves G, H and K. For H to be 0, we need to have $C = 5$ and $B+D = K$ (not $1K$). But $B+D+1 = 1G \geq 10$, so $K = 9$. But then $B+D = 10$ and we also have $G = 0$. Thus $H \neq 0$. **Assume next** $K = 0$. We will see that does not work. In fact, then $B+D = 10$ so $1G = B+D+1 = 11$ and $G = 1$. Notice that since AB is even, B must be even so we see that $\{B, D\} = \{2, 8\}$ or $\{B, D\} = \{4, 6\}$. Now $C \neq 5$; otherwise $H = 1 = G$. If $C = 9$ is also out; if $C = 9$, then $C+C+1 = 19$, but $C+C+1 = 1H$ so $H = 9 = C$; not possible. So $C = 6, 7$ or 8. Let us see now if we can decide which letter represents 9. Since $A+E < 10$, it can't be A nor E. It can't be B or D, since B, D are both even in this $K = 0$ model. We already saw that it can't be C. For H to be 9 we need $C+C+1 = 19$, but then $C = 9 = H$. So $H \neq 9$. The only letters not eliminated are F and L; since $F = L+1$ we conclude $L = 8, F = 9$. That $E = 8$ eliminates one of the choices for the pair B, D; we conclude $\{B, D\} = \{4, 6\}$. Recalling that C was one of $6, 7, 8$, we now see that $C = 7$. Then $1H = 7+7+1 = 15$ implies $H = 5$. We thus have $\{B, C, D, F, G, H, K, L\} = \{0, 1, 4, 5, 6, 7, 8, 9\}$ leaving only $2, 3$ for A, E. But $A+E = 8$, so this is not possible. **We conclude that K $\neq$ 0.**

Returning to where we were before assuming $K = 0$, this leaves G or H as being equal to 0. We can easily eliminate H. In fact, the only way H can be 0 is if $C = 5$ which was already ruled out. We have established that $G = 0$. This means that $B+D+1 = 10$ so $B+D = 9$; that is, $K = 9$. Since B is even, we have (B, D) is one of $(2, 7), (4, 5), (6, 3)$ or $(8, 1)$. We now **claim** that $C = 7$. The alternative is $C = 6$ or $C = 8$. Assume first $C = 6$. Then $H = 2$ and B must be one of $4, 8$. Suppose first $B = 4$. Then $D = 5$ and $\{B, C, D, G, H, K\} = \{0, 2, 4, 5, 6, 9\}$, leaving $\{A, E, F, L\} = \{1, 3, 7, 8\}$. Since $F = L+1$, this forces $L = 7, F = 8$; but then we get $7 = L = A+E = 1+3$, a contradiction. Assume next (still assuming $C = 6$)

that $B = 8$. Then $D = 1$ and we have to modify what we did above to $\{B, C, D, G, H, K\} = \{0, 1, 2, 6, 8, 9\}$, leaving $\{A, E, F, L\} = \{3, 4, 5, 7\}$. Using again $F = L + 1$, this would mean that $L = 4, F = 5$; then $A + E = 3 + 7 = 10$. which is ruled out. One establishessimilarly that $C = 8$ is not possible, proving with this that $C = 7$. Then $H = 4$ and since $B \neq H = 4$, $D \neq C = 7$, we either have $B = 6, D = 3$ or $B = 8, D = 1$. With the first choice the only choices left for A, E, F, L are $1, 2, 5, 8$; once again, $F = L + 1$ would force $L = 1, F = 2$, but then L cannot be $A + E$. So $B = 8$, $D = 1$. Our computation looks (so far) as follows

$$\begin{array}{cccccc} & \mathrm{A} & 8 & 7 & 1 & \mathrm{E} \\ + & \mathrm{E} & 1 & 7 & 8 & \mathrm{A} \\ \hline & \mathrm{F} & 0 & 4 & 9 & \mathrm{L} \end{array}$$

This forces $\{A, E, F, L\} = \{2, 3, 5, 6\}$. This is possible with $L = 5, F = 6$ and $\{A, E\} = \{2, 3\}$. If $A = 2$,then $AB = 28$, not twice a prime. If $A = 3$, then $AB = 38 = 2 \times 19$, twice a prime. The solution is

$$\begin{array}{cccccc} & 3 & 8 & 7 & 1 & 2 \\ + & 2 & 1 & 7 & 8 & 3 \\ \hline & 6 & 0 & 4 & 9 & 5 \end{array}$$

11. The integers 234, 417, 645 share a curious property: All three digits are different and one of the three digits is the average of the other two. How many three-digit numbers have this property? That is, how many three digit numbers are composed of three **distinct** digits such that one digit is the average of the other two?

    Solution. Let us first get all sets of three digits $\{a, b, c\}$ such that $c = (a + b)/2$,$a < b$. For c to be a digit, either a, b are both even (10 choices) or both odd (also 10 choices). We have 20 choices in all. Each one of these choices can be arranged in 6 different ways. For example, from $(3, 5, 4)$ we get the integers

    345, 354, 453, 435, 534, 543.

    This gives a total of $6 \times 20 = 120$ integers. But some of these will have a leading 0! If $a = 0$, then b is even, $c = b/2$, so we have to disregard the integers $021, 012, 042, 024, \ldots, 084, 084$, eight integers in all. The final answer is that there are 112 such integers.

12. Suppose that the number a satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$? (AMC 10A)
    Solution. The shortest solution uses is probably as follows:

$$\begin{aligned} 16 &= 4^2 = (a + a^{-1})^2 = a^2 + 2 + a^{-2}, \quad \text{hence} \quad a^2 + a^{-2} = 14. \\ 196 &= 14^2 = (a^2 + a^{-2})^2 = a^4 + 2 + a^{-4} \end{aligned}$$

so that $a^4 + a^{-4} = 194$.

One can also solve for a; multiplying the equation by a we get $a^2 - 4a + 1 = 0$ so that

$$a = \frac{4 \pm \sqrt{12}}{2} = 2 \pm \sqrt{3}.$$

Noticing that the roles of a, a^{-1} are symmetric and that $1/(2 + \sqrt{3}) = 2 - \sqrt{3}$, we can assume $a = 2 + \sqrt{3}$, then

$$\begin{aligned} a^2 &= 4 + 4\sqrt{3} + 3 = 7 + 4\sqrt{3} \\ a^4 &= 49 + 56\sqrt{3} + 48 = 97 + 56\sqrt{3}, \end{aligned}$$

$$a^{-4} = \frac{1}{97 + 56\sqrt{3}} = 97 - 56\sqrt{3}$$

so that $a^4 + a^{-4} = (97 + 56\sqrt{3}) + (97 - 56\sqrt{3}) = 194$.