# MATH CIRCLE AT FAU

## 09/29/2018
## Session # 2

**RULES**

- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.
- Except in case of an emergency, all smart devices are strictly forbidden. No smart phones, smart watches, or time machines. Only smart people.

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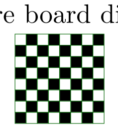
1. **The $n$-Queens Problem.**

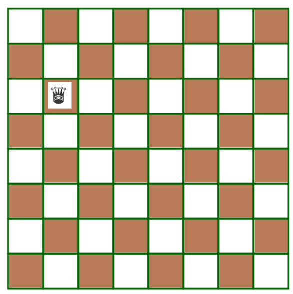
**You do NOT need to know how to play chess to work this problem!**

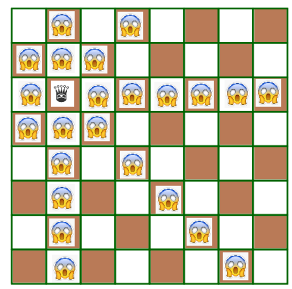
This is a classical problem; to look it up today on the internet would be cheating. Here is how it goes. A standard chess board is a square board divided into 64 squares, in 8 rows of 8 squares, in alternating colors (black and white, for example): 

All you need to know about the game of chess is that one of the pieces, a very powerful one, is the queen ♛. The queen can move on the board either diagonally or across rows or columns; any opponent chess piece in the path of the queen could be "eaten" (eliminated). The *eight queen problem* was first posed some hundred fifty years ago, here it is:

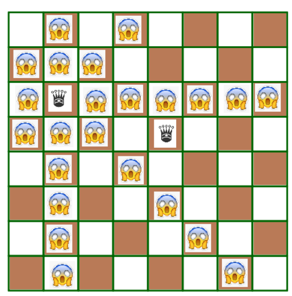
*Place 8 queens on the chessboard so that no queen threatens another queen; that is no two queens should be in the same row, column or diagonal.*

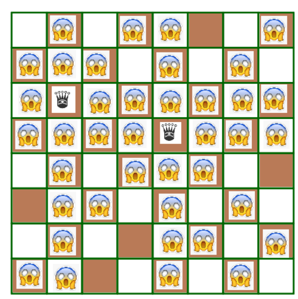
As an example, if I place a queen as in the following picture on the left, the picture on the right has an emoji in all the threatened squares:





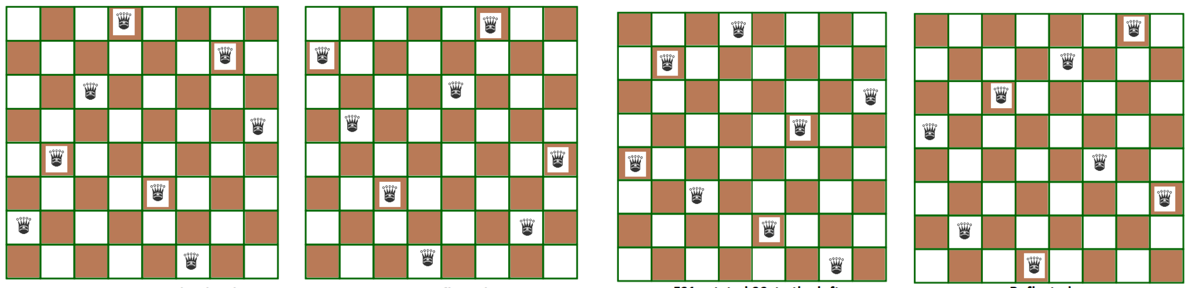
If I now place another queen in one of the free squares, more squares are threatened. For example, if I place a queen as in the picture on the left below, the threatened squares are shown on the right.



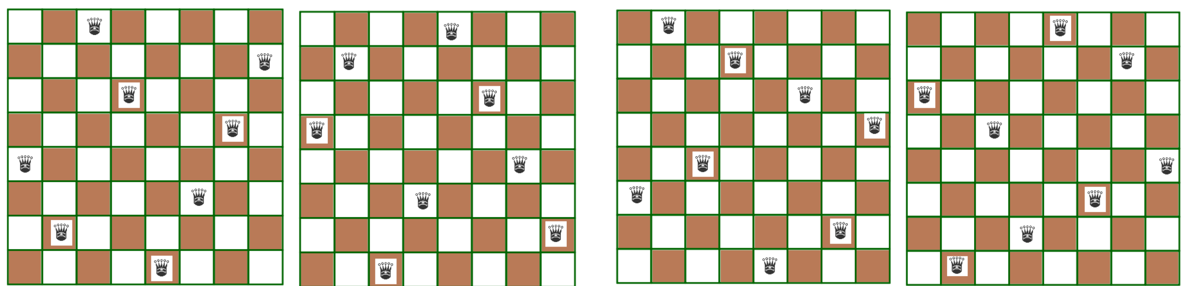


It has been determined (proved) that there are 12 so called fundamental solutions of the eight queens problem, and all others (to a total of 92!) can be obtained from one of these twelve by either rotating the board or reflecting. The next page shows one of the fundamental solutions, and 8 others you can get from it by rotating and reflecting. **Can you find other solutions?** Maybe you should start with smaller boards. It is obviously NOT possible to place two queens on a $2 \times 2$ board so they don't threaten each other. It is also not hard to see that it is it is NOT possible to place three queens on a $3 \times 3$ board without one of them threatening another one. What about placing four queens on a $4 \times 4$ board? Five on a $5 \times 5$ board. There are solutions for placing $n$ queens on an $n \times n$ board once $n$ is greater than 3. But as $n$ increases, one knows less and less about how many solutions there are. For boards larger than $27 \times 27$ not much is known.

Here are 8 solutions of the 8 queens problem. The first one is fundamental. The others are obtained either by rotation (as indicated), or by reflection across the horizontal middle line of the board to its left.



A fundamental solution (FS1) | FS1 reflected | FS1 rotated 90 to the left | Reflected



FS1 rotated 180° to the left | Reflected | FS1 rotated 270° to the left | Reflected

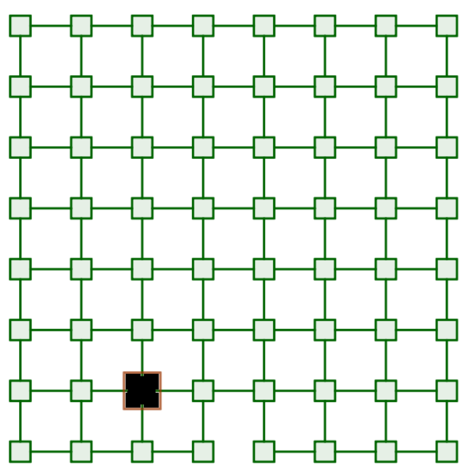
(a) Can you find other solutions?

(b) How **many** solutions are there for the $4 \times 4$, the $5 \times 5$, the $6 \times 6$, and the $7 \times 7$ puzzle?

(c) Can one place **9** queens on the $8 \times 8$ board so none threatens any other? WHY?

(d) Can you find solutions when $n \geq 9$?

2. This is another old puzzle. The picture below is a map with each little square representing a town; 64 towns in all. The lines connecting them are roads. **Notice there is no direct road connecting the fourth town of the bottom row with the fifth town of the same road.** The puzzle consists in starting from the large black town and visit all other towns once and only once in **fifteen** straight trips.

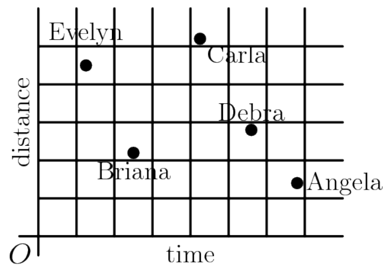


3. Connie multiplies a number by 2 and gets 60 as her answer. However, she should have divided the number by 2 to get the correct answer. What is the correct answer? (AMC 8)

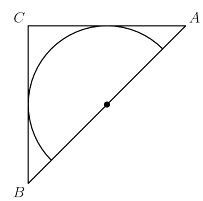
4. Bill walks $\frac{1}{2}$ mile south, then $\frac{3}{4}$ mile east, and finally $\frac{1}{2}$ mile south. How many miles is he, in a direct line, from his starting point? Your answer (in miles) should be in the from $\frac{a}{b}$, where $a, b$ are integers with no common divisors other than 1.(AMC 8)

5. How many different isosceles triangles have integer side lengths and perimeter 23? (AMC 8)

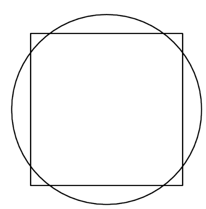
6. The results of a cross-country team's training run are graphed below. Which student has the greatest average speed? (AMC 8)



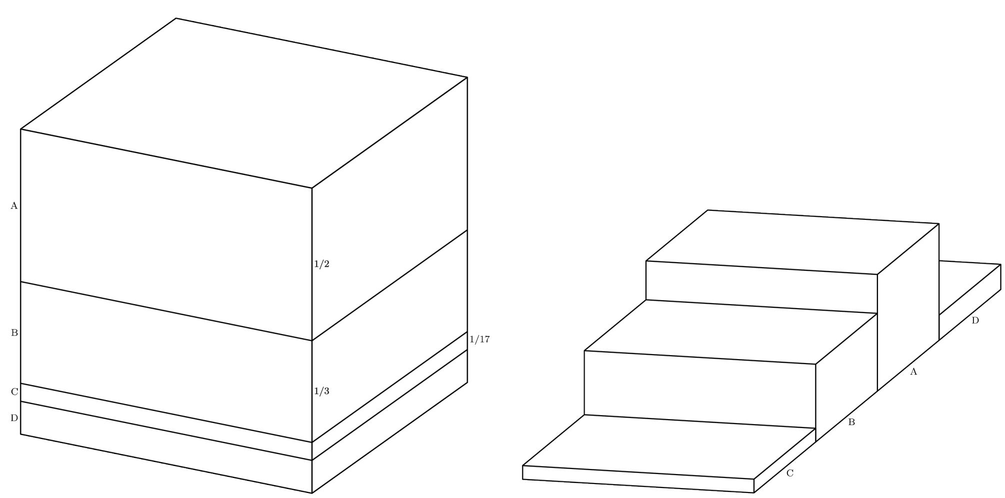

7. Isosceles right triangle $ABC$ encloses a semicircle of area $2\pi$. The circle has its center $O$ on hypotenuse $\overline{AB}$ and is tangent to sides $\overline{AC}$ and $\overline{BC}$. What is the area of triangle $ABC$? (AMC 8)




8. A square with side length 2 and a circle share the same center. The total area of the regions that are inside the circle and outside the square is equal to the total area of the regions that are outside the circle and inside the square. What is the radius of the circle? (AMC 8)



9. A one-cubic-foot cube is cut into four pieces by three cuts parallel to the top face of the cube. The first cube is $\frac{1}{2}$ foot from the top face. The second cut is $\frac{1}{3}$ foot below the first cut, and the third cut is $\frac{1}{17}$ foot below the second cut. From the top to the bottom the pieces are labeled A, B, C, and D. The pieces are then glued together end to end as shown in the second diagram. What is the total surface area of this solid in square feet? (AMC 8)



10. In the computation below, A, B, C, D, E, F, G, H, K, L are distinct (so no two are equal) digits and AB is double a prime number. Find A,B,. . . , H,K,L.

$$\begin{array}{cccccc} & A & B & C & D & E \\ + & E & D & C & B & A \\ \hline & F & G & H & K & L \end{array}$$

11. The integers 234, 417, 645 share a curious property: All three digits are different and one of the three digits is the average of the other two. How many three-digit numbers have this property? That is, how many three digit numbers are composed of three **distinct** digits such that one digit is the average of the other two?

12. Suppose that the number $a$ satisfies the equation $4 = a + a^{-1}$. What is the value of $a^4 + a^{-4}$? (AMC 10A)