Math Circle, SOLUTIONS for 9/22/18

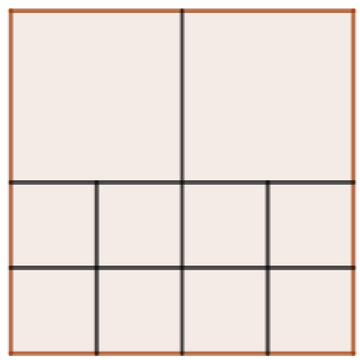
1. **The Mondrian Problem**

   A lot of information can be found on the web on this problem. I am leaving it at that.

2. A square with an integer side length is cut into 10 squares, all of which have integer side length and at least 8 of which have area 1. What is the smallest possible value of the length of the side of the original square? (AMC 8-2012) Your choices are:

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution. We have 8 squares of area 1 and two additional squares of unknown area (could also be 1). The 8 squares that we know have area 1 add up to area 8. Since the total area must be a square we are looking for two squares such that when their sum is added to 8 we get a square. The easiest (and smallest) resolution is two squares of side length 2: $8 + 2^2 + 2^2 = 16 = 4^2$. The answer is $\boxed{4}$.



3. In a town of 351 adults, every adult owns a car, a motorcycle, or both. If 331 adults own cars, and 45 adults own motorcycles, how many of the car owners do **not** own a motorcycle. (AMC 8-2011)

   Solution. If we add the total number of cars and motorcycles owned we get 376 vehicles. This overshoots the total population by $376 - 351 = 25$ because we counted twice the people who own both cars and motorcycles. This means that there are 25 people owning both cars and motorcycles. The number of people owning cars but not motorcycles is thus $331 - 25 = \boxed{306.}$.

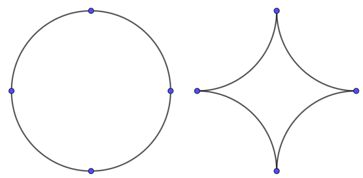
4. In a jar of red, blue, and green marbles, all but 6 are red marbles, all but 8 are green, and all but 4 are blue. How many marbles are in the jar? (AMC 8-2012)

   Solution. Let r, b, g be the number of red, blue, and green marbles, respectively. What we are told can be expressed by the following equations:

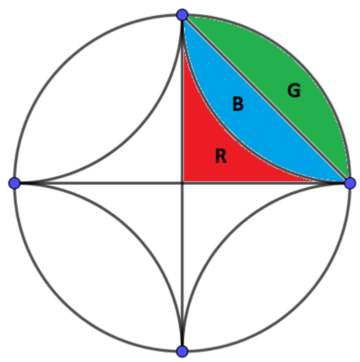
$$b + g = 6, \quad b + r = 8, \quad r + g = 4.$$

   Let's do some trial and error. From the first equation, $b \leq 6$. If $b = 6$, then $g = 4$; then the last equation says $r = 0$ and there is no way the middle equation holds. If $b = 5$ we similarly get $g = 1$, $r = 3$, and it works! We should really check that this is the only way it can work. If $b = 4$, we get $g = 2$, $r = 2$, and the middle equation can't hold. Similarly we see that $b = 3, 2, 1, 0$ can't work. The answer is $5 + 3 + 1 = \boxed{9.}$

5. A circle of radius 2 is cut into four congruent arcs. The four arcs are arranged to form the star figure as shown. What is the area of the star figure? (AMC 8-2012)



Solution. Suppose the circle has radius r. In the figure below the star is placed inside the circle.

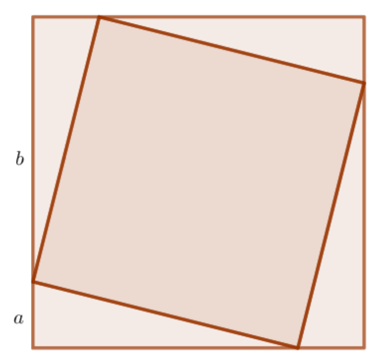


Let R be the red area, B the blue area and G the green area. Then $R+B+G$ equals one fourth of the circle; that is, $R+B+G=\pi r^2/4$. Now G is the quarter circle minus the isosceles right triangle $R+B$ of legs equal to r; that is $G=\frac{1}{4}\pi r^2-\frac{1}{2}r^2$. But $B=G$ and, as said before, $R+B=\frac{1}{2}r^2$. This tells us that

$$R=\frac{1}{2}r^2-\left(\frac{1}{4}\pi r^2-\frac{1}{2}r^2\right)=r^2-\frac{1}{4}\pi^2.$$

The total area of the star is 4 times the red area, so the final answer is (given that $r=2$) $\boxed{4(4-\pi).}$

6. A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments,one of length a, the other one of length b. What is the value of ab? (The picture is NOT drawn to scale) (AMC 8-2012)



Solution. The area of the larger square equals the sum of the areas of four right triangles of legs a, b and the area of the smaller square. Thus

$$5=4\times\frac{1}{2}ab+4$$

so that $\boxed{ab=\frac{1}{2}.}$

7. A fair six-sided die is rolled twice.  What is the probability that the first number that comes up is greater than or equal the second number? (AMC 8-2012)

   Solution. Let us write (a, b) to indicate that the first number showing up was a, the second b. For example, $(3, 5)$ indicates that the first number is 3, the second 5. There are then $6 \times 6 = 36$ possible outcomes. Of these, our favorable outcomes are those of the form (a, b) with $a \geq b$ It is easy to list all of these outcomes:

   $$(1,1), (2,1), (2,2), (3,1), (3,2), (3,3), (4,1), \ldots (6,6).$$

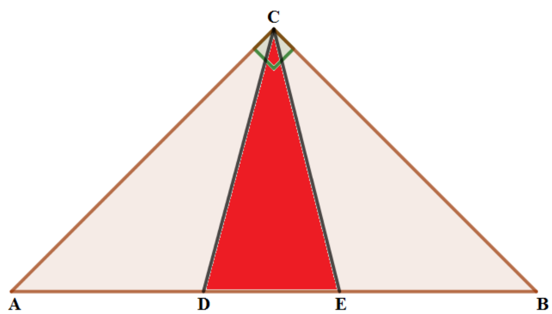
   There is 1 with first number 1, there are 2 with first number 2, 3 with first number 3, and so forth, to 6 with first number 6; a total of $1+2+3+4+5+6 = 21$ such outcomes. The answer is $\dfrac{21}{36} = \boxed{\dfrac{7}{9}.}$

8. How many digits are there in the product $4^5 \cdot 5^{10}$

   Solution. Since $4^5 5^{10} = 2^{10} 5^{10} = 10^{10} = 10,000,000,000$ so the answer is $\boxed{11.}$

Harder Problems

1. The isosceles right triangle has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$ (AMC 10A-2015)



   Note: To solve this problem you probably need to know some trigonometry. Your answer should have the form $\dfrac{a + b\sqrt{c}}{d}$ where a, b, d are integers (not necessarily all positive) with no common divisor other than 1 and c is a square free positive integer.

   Solution. We have

   $$12.5 = [ABC] = \frac{1}{2} AC \cdot BC = \frac{1}{2} AC^2$$

   so that $AC = BC = 5$. Let h be the altitude from C of the triangle, then $h = 5/\sqrt{2}$. We will find $[ADC]$ the area of $\triangle ADC$. We have

   $$[ADC] = \frac{1}{2} hAD = \frac{5}{2\sqrt{2}} AD = \frac{5\sqrt{2}}{4} AD.$$

   To find AD we use the law of sines in the triangle ADC. In this triangle, the angle opposite to AD measures $90/3 = 30°$, the angle opposite to AC measure (as is easy to figure out) $105°$. We thus have

   $$\frac{AD}{\sin 30°} = \frac{AC}{\sin 105°}; \text{ using } AC = 5, \sin 30° = \tfrac{1}{2}, \text{ thus } AD = \frac{5}{2 \sin 105°}.$$

   Now, a diagram show that $\sin 105° = \sin 75°$ so that

   $$\sin 105° = \sin 75° = \sin(30° + 45°) = \sin 30° \cos 45° + \cos 30° \sin 45° = \frac{1}{2}\frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2} + \sqrt{6}}{4}.$$

Therefore,

$$AD = \frac{10}{\sqrt{6}+\sqrt{2}} = \frac{5(\sqrt{6}-\sqrt{2})}{2}$$

and

$$[ADC] = \frac{5\sqrt{2}}{4}\frac{5(\sqrt{6}-\sqrt{2})}{2} = \frac{25(\sqrt{3}-1)}{4}.$$

Finally

$$[CDE] = [ABC] - 2[ADC] = 12.5 - \frac{25(\sqrt{3}-1)}{2} = \frac{25}{2} - \frac{25(\sqrt{3}-1)}{2} = \boxed{\frac{50-25\sqrt{3}}{2}.}$$

2. The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all the possible values of a? (AMC 10A-2015)

   Solution. Let p, q be the roots of the equation, then $p + q = a, pq = 2a$, thus $2p + 2q = pq$. If $q = 2$ we get the impossible equation $2p + 4 = 2p$, so $q \neq 2$. Then $p = \dfrac{2q}{q-2}$. Since p is an integer, this implies that $q - 2$ divides $2q$, so for some integer k,

$$2q = k(q-2). \tag{1}$$

   Suppose m is an odd prime divisor of $q-2$; then (by (1) $m|2q$, hence $m|q$. But if m divides both q and $q-2$, it would also divide 2, which is impossible, Thus $q - 2$ has no odd prime divisors, implying $q - 2 = 2^r\epsilon$ for some non-negative integer r, $\epsilon = \pm 1$. Thus (1) becomes

$$4 + 2^{r+1}\epsilon = 2^r\epsilon k.$$

   The right hand side of this equation is divisible by 2^r; so must the right hand side be, implying that $2^r|4$. Thus $r = 0, 1$, or 2. Recalling $q = 2 + 2^r\epsilon$ we get the following possibilities for q with $r = 0, 1, 2$, $\epsilon = \pm 1$:

$$q = 1,\ 2,\ 3,\ 4,\ 6,\ -2.$$

   Eliminating the non-possible value $q = 2$, we see that all the other values give integer values of p and we get the following pairings of integers (q, p):

$$(1, -2),\ (3, 6),\ (4, 4),\ (6, 3),\ (-2, 1).$$

   The corresponding list of possible a values is -1, 9, 8 adding to $\boxed{16}$.

3. We want to consider here all possible pairs of real numbers a and b for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0,$$

   has at least one real root. What is the minimum possible value of $a^2 + b^2$ for all such pairs?

   Some comments. If $a^2 + b^2 = 0$, then $a = b = 0$ and the equation is $x^4 + 1 = 0$, which has no real roots. So the minimum must be > 0. On the other hand, if $a = -4, b = 6$, the equation is

$$0 = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4,$$

   which has the real solution 1, so the minimum is $\leq (-4)^2 + 6^2 = 52$. This problem could be lengthy and you might want to work on it at home. If you get a solution you can email it to me and I will let you know if you are right.

   Solution. (One of many ways of solving it) It is easy to see that if α is a root of the equation, so is $1/\alpha$. Moreover, if w is a complex root of the equation, so is its conjugate $\bar{w}$. Assuming then that the equation has at least one real root, then the roots of the equation are of the form $\alpha, 1/\alpha, \beta, 1/\beta$ where α is real. We are not assuming all these roots are distinct. If β is not real, then, since $\bar{b}$ is also a root, we must have $\bar{\beta} = 1\beta$, thus $|\beta| = 1$. Then setting

$$p = \alpha + \frac{1}{\alpha}, \qquad q = \beta + \frac{1}{\beta}$$

we get

$$x^4+ax^3+bx^2+ax+1 = (x-\alpha)(x-\frac{1}{\alpha})(x-\beta)(x-\frac{1}{\beta}) = (x^2-px+1)(x^2-qx+1) = x^4-(p+q)x^3+(pq+2)x^2+1$$

so that $a = -(p+q)$, $b = pq+2$. Notice that q is real; this is clear if β is real, otherwise $q = \beta + \bar{\beta} = 2\text{Re }\beta$. We now have

$$a^2+b^2 = (p+q)^2 + (pq+2)^2 = p^2+q^2+p^2q^2+6pq+4 = (p^2+1)\left(q+\frac{3p}{p^2+1}\right)^2 + \frac{(p^2-2)^2}{p^2+1}. \quad (2)$$

It also not hard to see that for α real on has $|p| = \left|\alpha + \frac{1}{\alpha}\right| \geq 2$ (with 2 achieved for $\alpha = 1$ or -1. This is easily seen using calculus, but it is also easy without calculus: It suffices to prove it for $\alpha > 0$. In this case

$$\alpha + \frac{1}{\alpha} - 2 = \frac{(\alpha-1)^2}{\alpha} \geq 0.$$

By (2).

$$a^2+b^2 \geq \frac{(p^2-2)^2}{p^2+1};$$

the right hand side of this last inequality increases for $p \geq \sqrt{2}$; since $p \geq 2$ we proved

$$a^2+b^2 \geq \left.\frac{(p^2-2)^2}{p^2+1}\right|_{p=2} = \frac{4}{5}.$$

It remains to be seen that this value can be reached. By (2), for $a^2+b^2 = 4/5$ it is necessary and sufficient that $q = -\frac{3p}{p^2+1}$ for $p=2$; i.e., $q = -6/5$. Since $6/5 < 2$, this precludes the possibility of $q = \beta + (1/\beta)$ with β real; we must have $q = 2\text{Re }\beta$ for some β of absolute value 1. This is possible, for example $\beta = -\frac{3}{5} + i\frac{4}{5}$. The final answer is that the minimum value of a^2+b^2 for which the equation in question has a real root is $\boxed{\frac{4}{5}.}$ In fact, the equation

$$x^4 - \frac{4}{5}x^3 - \frac{2}{5}x^2 - \frac{4}{5}x + 1 = 0$$

has the roots $1, 1, -\frac{3}{5}+\frac{4}{5}i, -\frac{3}{5}-\frac{4}{5}i$ and satisfies $\left(-\frac{4}{5}\right)^2 + \left(\frac{2}{5}\right)^2 = \frac{4}{5}$.