MATH CIRCLE AT FAU

09/22/2018

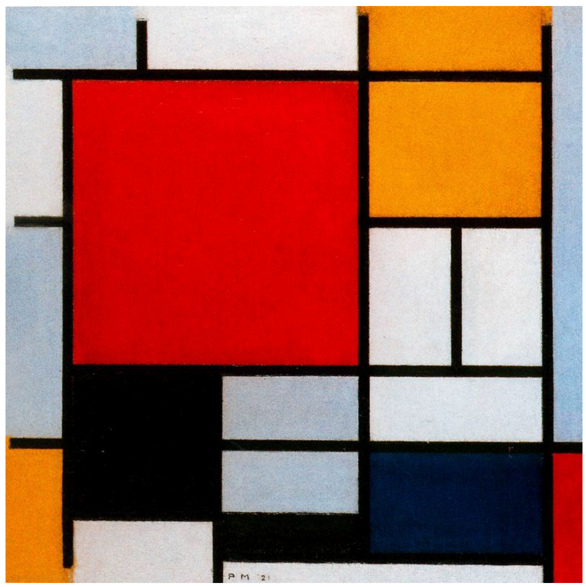
Session # 1

RULES

- Work the problems in any order. Some problems are harder than others; do what you can. **If later on you take the time to figure out how to solve the problems you could not solve, you will learn more from what you could not do, than from what you could do.**
- If you think you have finished a problem correctly, tell one of the organizers. If it is really correct, he or she will certify that it is correct.
- Don't feel shy about asking for hints.
- Don't feel shy about getting up, walking around, or talking with anybody you want to talk to.
- If you want to write on one of the whiteboards, we have markers available.
- **In this session, all smart devices are strictly forbidden. No smart phones, smart watches, or time machines.** There is a reason for this

1 The Mondrian Problem

Piet Mondrian  was a Dutch painter, considered one of the great artists of the 20th Century. Some of his favorite paintings consist of a square or rectangle divided into non-overlapping rectangles, like this one called *Composition With Large Red Plane Yellow Black Gray And Blue*, painted in 1921.



So here is the problem. We imagine that Mondrian has been captured by evil aliens (maybe Martians, but most likely Venusians) who force him to paint picture after picture under the following conditions:

1. Mondrian must cover the canvas with rectangles that ouch only (if at all) on their edges. The canvas is a square of integer sides (2×2 or 3×3, or 4×4, or $\ldots 1000 \times 1000$). All rectangles are to have integer sides.

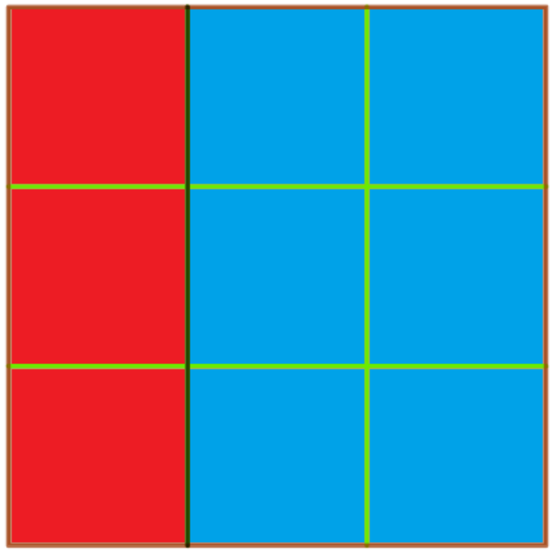
2. Every rectangle on the canvass must be different. . . so Mondrian cannot paint both a 4×5 and a 5×4 rectangle. However, he can paint a 4×5 and 2×10 rectangle.

3. And here is the main thing!: Mondrian must try to minimize his score. The score of a painting is the area of the largest rectangle minus the area of the smallest rectangle.

4. Some of the nastier aliens also imposed one additional condition, when coloring, Mondrian must use as few colors as possible, colors cannot touch along edges or corners.

Your mission, if you decide to accept it, is to help Mondrian. The aliens have promised that if he can get squares of sizes 3×3, 4×4, 5×5, and a lot more (how many they refuse to tell him) divided into rectangles following the rules so that the score is as low as possible for each size of square, they will let him go. Turn the page for examples.

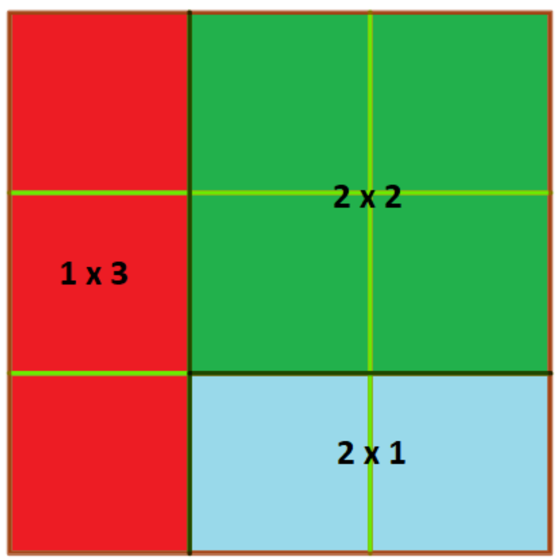
The 3×3 square has been partitioned into a 1×3 rectangle of area 3 and a 2×3 rectangle of area 6.



The score is $6 - 3 = 3$. Is this best possible?

NO!

Here we have the same 3×3 square partitioned into 3 rectangles of areas 2, 3, 4.



The score is $4 - 2 = 2$. Is this best possible?

We will have sheets of graphed paper available for you to try. Let's see who comes up with the best scores!

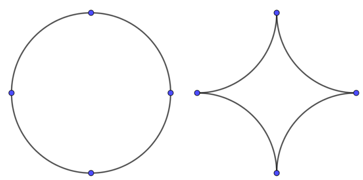
2. A square with an integer side length is cut into 10 squares, all of which have integer side length and at least 8 of which have area 1. What is the smallest possible value of the length of the side of the original square? (AMC 8-2012) Your choices are:

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

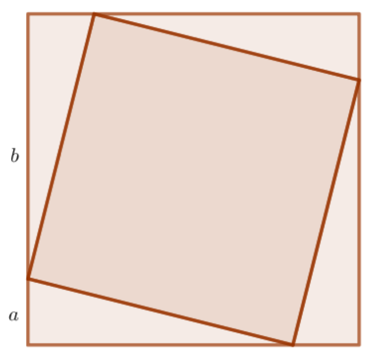
3. In a town of 351 adults, every adult owns a car, a motorcycle, or both. If 331 adults own cars, and 45 adults own motorcycles, how many of the car owners do **not** own a motorcycle. (AMC 8-2011)

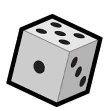
4. In a jar of red, blue, and green marbles, all but 6 are red marbles, all but 8 are green, and all but 4 are blue. How many marbles are in the jar? (AMC 8-2012)

5. A circle of radius 2 is cut into four congruent arcs. The four arcs are arranged to form the star figure as shown. What is the area of the star figure? (AMC 8-2012)



6. A square with area 4 is inscribed in a square with area 5, with one vertex of the smaller square on each side of the larger square. A vertex of the smaller square divides a side of the larger square into two segments,one of length a, the other one of length b. What is the value of ab? (The picture is NOT drawn to scale) (AMC 8-2012)

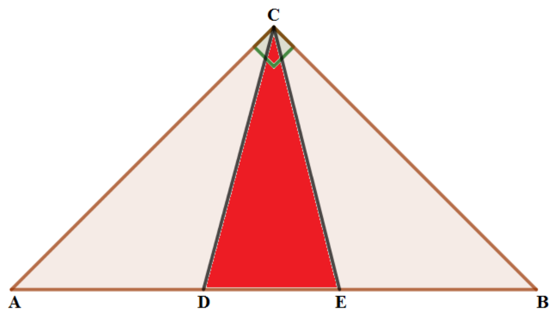


7. A fair six-sided die is rolled twice.  What is the probability that the first number that comes up is greater than or equal the second number? (AMC 8-2012)

8. How many digits are there in the product $4^5 \cdot 5^{10}$

Harder Problems

1. The isosceles right triangle has right angle at C and area 12.5. The rays trisecting $\angle ACB$ intersect AB at D and E. What is the area of $\triangle CDE$ (AMC 10A-2015)



Note: To solve this problem you probably need to know some trigonometry. Your answer should have the form $\dfrac{a+b\sqrt{c}}{d}$ where a, b, d are integers (not necessarily all positive) with no common divisor other than 1 and c is a square free positive integer.

2. The zeros of the function $f(x) = x^2 - ax + 2a$ are integers. What is the sum of all the possible values of a? (AMC 10A-2015)

3. We want to consider here all possible pairs of real numbers a and b for which the equation

$$x^4 + ax^3 + bx^2 + ax + 1 = 0,$$

has at least one real root. What is the minimum possible value of $a^2 + b^2$ for all such pairs?

Some comments. If $a^2 + b^2 = 0$, then $a = b = 0$ and the equation is $x^4 + 1 = 0$, which has no real roots. So the minimum must be > 0. On the other hand, if $a = -4, b = 6$, the equation is

$$0 = x^4 - 4x^3 + 6x^2 - 4x + 1 = (x-1)^4,$$

which has the real solution 1, so the minimum is $\leq (-4)^2 + 6^2 = 52$. This problem could be lengthy and you might want to work on it at home. If you get a solution you can email it to me and I will let you know if you are right.