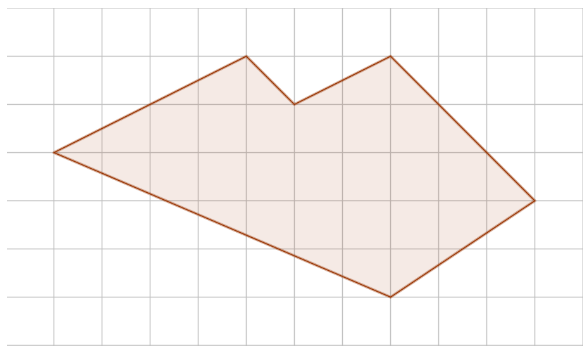
FAU Math Circle
12/5/2015

THE END OF YEAR 2015 COMPETITION
SOLUTIONS

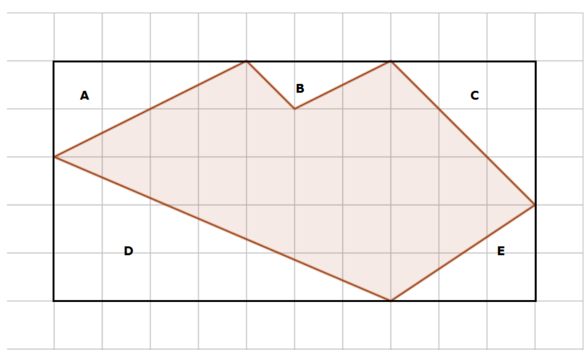
1. Erika promised to sell an average of 20 boxes of girl scout cookies per week over a period of six weeks. In the first five weeks she sold 15, 25, 18, 19, and 20 boxes. How many boxes does she have to sell in the sixth week to keep her promise?

   Solution. In the first five weeks she sold $15+25+18+19+20 = 97$ boxes. She should have sold $5\times 200 = 100$ boxes, soshe is short 3 boxes. That means that in the sixth week she has to sell 23 boxes to make up for the deficit.

2. Find the area of the figure in the following diagram, if the areas of the squares is 1.



   Solution. Possibly the easiest way is to see that the figure can be enclosed in a 10×5 rectangle, add up the areas of the triangles marked as A, B, C, D, E in the picture below, and subtract that from the area of the rectangle.



   The areas are, using the same letter to indicate the area as the triangle: $A = \frac{1}{2}(2\times 4) = 4$, $B = \frac{1}{2}(3\times 1) = 1.5$, $C = \frac{1}{2}(3\times 3) = 4.5$, $D = \frac{1}{2}(7\times 3) = 10.5$, $E = \frac{1}{2}(3\times 2) = 3$. The area of the figure is

$$(10\times 5) - (4 + 1.5 + 4.5 + 10.5 + 3) = \boxed{26.5}$$

3. Here is a puzzle that is over 1700 years old:

   Demochares (just call him D) has lived a fourth of his life as a boy, a fifth as a youth, a third as a man, and 13 years as an old man (people got old at a young age in those far away days). How old is he?

Solution. There is a cheating solution. Since the answer will quite likely be a whole number, it is likely a number whose fourth, fifth and third are whole numbers, thus a number divisible by $4 \times 3 \times 5 = 60$. The most likely such number is 60 itself. One fourth of 60 is 15, one fifth is 12 and one third is 20; $15 + 12 + 20 = 47$ and since $47 + 13 = 60$ we hit the jackpot. The answer is 60 years.

For a better, more rigorous answer, let x be D's age. Then

$$\frac{1}{4}x + \frac{1}{5}x + \frac{1}{3}x + 13 = x.$$

Now

$$\frac{1}{4}x + \frac{1}{5}x + \frac{1}{3}x + 13 = \left(\frac{1}{4} + \frac{1}{5} + \frac{1}{3}\right)x + 13 = \frac{15 + 12 + 20}{60}x + 13 = \frac{47}{60}x + 13$$

so that

$$x = \frac{47}{60}x + 13 \quad \text{so} \quad \left(1 - \frac{47}{60}\right)x = 13$$

which is the same as $\frac{13}{60}x = 13$ or $x = 60$.

4. In an election 5219 votes were cast for four candidates: Amy, Beth, Meg and Jo. Amy won getting 22 more votes than Beth, 30 more votes than Meg, and 73 more votes than Jo. How many votes did each candidate get?

Solution. Let us write down what we know about the votes. If we call a the amount of votes obtained by Amy, then we know:

Candidate	Number of votes
Amy	a
Beth	$a - 22$
Meg	$a - 30$
Jo	$a - 73$

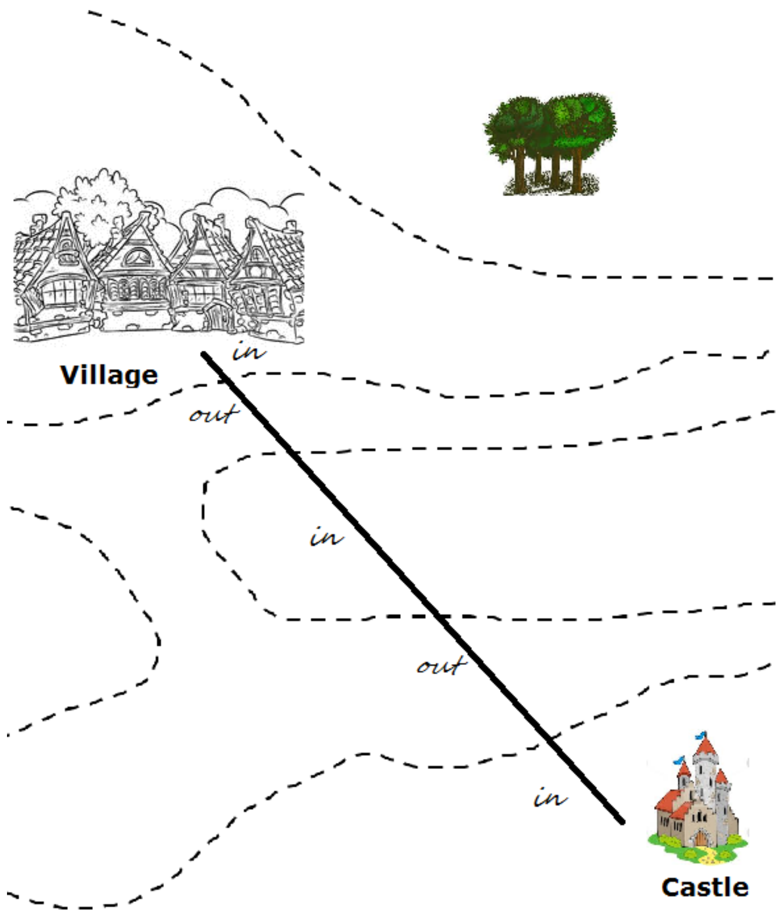
Adding up all the numbers of votes we get

$$a + a - 22 + a - 30 + a - 73 = 4a - (22 + 30 + 73) = 4a - 125.$$

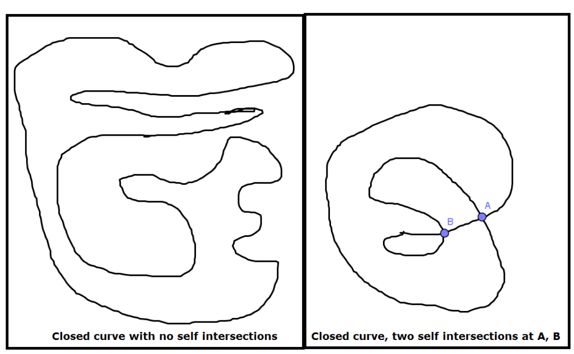
So we must have $4a - 125 = 5219$, so $4a = 5219 + 125 = 5344$ so $a = 5344/4 = 1336$. So the answer is

$$\boxed{\text{Amy: } 1336, \quad \text{Beth: } 1314, \quad \text{Meg: } 1306, \quad \text{Jo: } 1263.}$$

5. Baron Münchhausen once built a fence around his lands and marked it on a map. The fence is long gone, and now the Baron cannot remember whether or not the village of Hausenhoff is part of his possessions. Fortunately he has been able to find a fragment of the map containing his castle and the village (see figure below). He knows that the fence was shown on the map as a closed dashed curve without self intersections. Is the village on the Baron's land? **You must explain how you obtained your answer.**

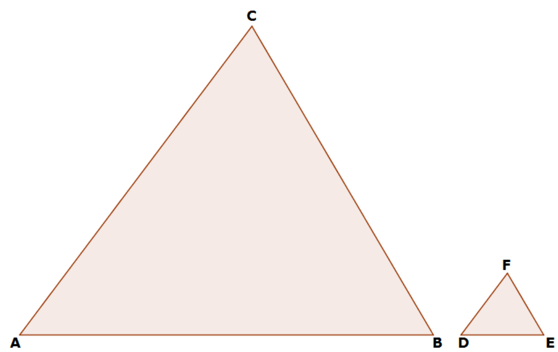


In case you wonder what is meant by "closed curve without self intersections, here are two examples:



Solution. The picture appears here with a line from the castle to the village. When we have a closed curve in the plane that does not intersect itself, every time we cross the curve if we were inside, we are out, if out, inside. So the answer is $\boxed{\textbf{YES.}}$

6. Triangle ABC is similar to triangle DEF. If $|DE| = 15$, $|DF| = 14$, and $|EF| = 13$, and the area of triangle ABC is 24,276 square feet, what are the lengths of AB, AC, and BC? Can our friend Heron be of any help?



NOTE: The figures are **NOT** drawn to scale. If drawn accurately, $\triangle ABC$ would have to be much, much larger. Or $\triangle DEF$ much, much smaller.

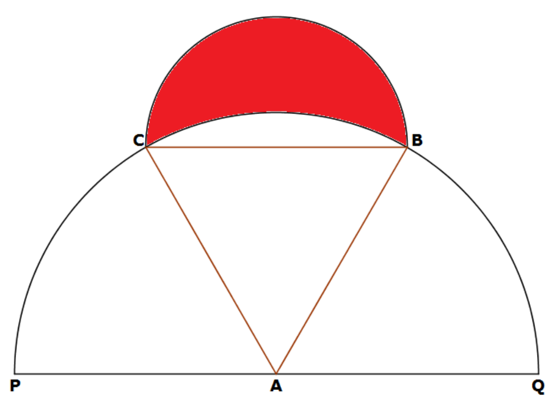
Solution. Similar triangles have proportional sides, and if the sides are in proportion x, the areas are in proportion x^2. So, for example, if we double the length of all sides of a triangle, the area gets multiplied by four, if we multiply the lengths of all sides by 3, the area gets multiplied by 9; and so forth. With this in mind we calculate the area of the small triangle, which is best done using Heron's formula (mentioned a few times in our circle): If a triangle has sides of lengths a, b, c then its are is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = (a+b+c)/2$ is the semi-perimeter. In our case $s = (15+14+13)/2 = 21$ and

$$A = \sqrt{21 \cdot 6 \cdot 7 \cdot 8} = \sqrt{3 \cdot 7 \cdot 2 \cdot 3 \cdot 7 \cdot 2 \cdot 2 \cdot 2} = \sqrt{7^2 \cdot 3^2 \cdot 2^4} = 7 \cdot 3 \cdot 4 = 84.$$

The areas are thus in the ratio $24276/84 = 289 = 17^2$. The ratio between corresponding sides is therefore 17 and the sought for lengths are

$$\boxed{|AB| = 17 \cdot 15 = 255, \quad |AC| = 17 \cdot 14 = 238, \quad |BC| = 17 \cdot 13 = 221.}$$

7. The triangle ABC is an equilateral triangle having its vertex A at the midpoint of the diameter PQ of a halfcircle; its vertices B, C on the halfcircle. Another halfcircle is drawn with diameter BC. Find the area of the moonlike (crescent) region, called a *lune*, which is French for *moon*, shaded in red, knowing that $|PQ| = 4$ units. **Warning:** π and at least one square root will appear in the answer.



Solution. An equilateral triangle has all of its angles equal to $60 = 360/6$ degrees so that the circular sector bounded by AC and AB is one sixth of the full circle. The full circle has a radius of $|PQ|/2 = 2$ units, so the circular sector in question has an area of $\pi 2^2/6 = \dfrac{2}{3}\pi$ square units. The sides of $\triangle ABC$ are equal to the radius of the circle; the area of an equilateral triangle of sides of length 2 works out to $\sqrt{3}$. The circular segment above the secant BC has an area equal to the area of the circular sector minus the area of the triangle, thus an area of $\frac{2}{3}\pi - \sqrt{3}$ square units. The area of the lune equals the area of the smaller halfcircle minus the area of this circular segment. The smaller halfcircle has a diameter of 2 units, thus a radius of 1 unit, thus an area of $\pi/2$ square units. The answer is

$$\frac{\pi}{2} - (\frac{2}{3}\pi - \sqrt{3}) = \boxed{\sqrt{3} - \frac{\pi}{6} \text{ square units.}}$$

8. A positive integer plus its cube equals 592788. What is the integer? It may help to know that

$$592788 = 2 \times 2 \times 3 \times 7 \times 7057.$$

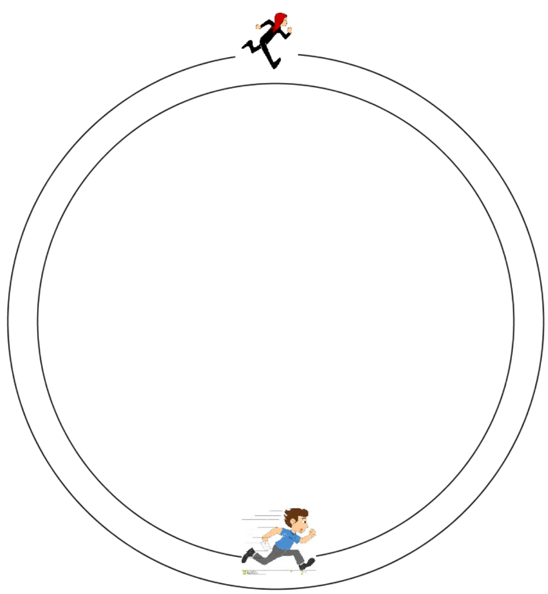
All of these factors are prime factors.

Solution. Let us call the integer we try to find x. Then we know that $x^3 + x = 592788$ which can be written in the form

$$x(x^2 + 1) = 2 \times 2 \times 3 \times 7 \times 7057.$$

Because 7057 is prime, it must divide either x or $x^2 + 1$. A bit of reflection shows it cannot be x, then $x^2 + 1$ would be less than $2 \times 3 \times 7 = 42$, so much less than $x^2 + 1$. With 7057 dividing $x^2 + 1$, x must be a product of some or all of the remaining factors; this leaves 12 possibilities for x. However x must be a pretty large number so that makes testing cases easier; we begin with the largest possibility, namely $7 \cdot 3 \cdot 2 \cdot 2 = 84$ and work our way downwards if necessary. If $x = 84$ then $x^3 + x = 592788$ and we are done. The answer is $\boxed{84.}$

9. Jack and Jill run in opposite directions on a circular track, starting at diametrically opposite points, at the same time.They first meet after Jill has run 300 feet. They next meet after Jack has run 450 feet past their first meeting point. Both run at a constant speed. What is the length of the track in feet?



Solution. I'll give two solutions.

Solution 1. When Jack and Jill meet first, the combined distances they ran equals half the length of the track. On meeting for the second time, their combined distances equal the full length of the track. So between the first and second meeting, both Jack and Jill must have covered twice the distance covered between their starting the run and their first meeting. So between the first and second meeting Jill ran $2 \times 300 = 600$ feet. Adding the distance run by Jack, namely 450 feet, we get the length of the track; $\boxed{\text{1050 feet.}}$

Solution 2. Let L be the length of the track and let v be Jack's speed, w Jill's speed. At their first meeting, Jill ran 300 feet; since they were on diametrically opposite points to start with, Jack ran $(L/2) - 300$ feet in that time. Thus

$$\frac{v}{w} = \frac{(L/2) - 300}{300} = \frac{L - 600}{600}.$$

At the next meeting, Jack ran 450 feet, thus Jill ran $L - 450$ feet. Thus

$$\frac{v}{W} = \frac{450}{L - 450}.$$

Equating the two expressions for v/w we get

$$\frac{L - 600}{600} = \frac{450}{L - 450}$$

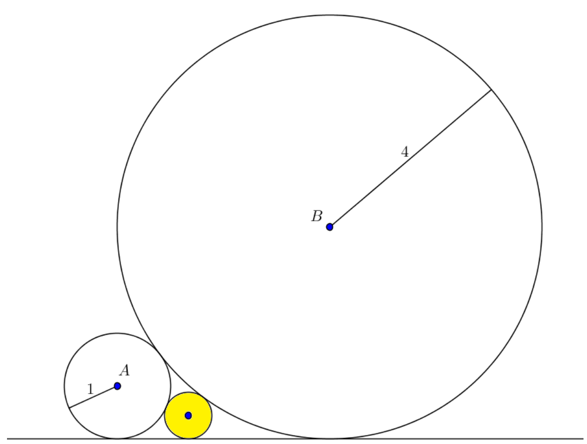
, which can be multiplied out to $(L - 600)(L - 450) = 600 \cdot 450$, thus $L^2 - 1050L = 0$. The solutions to this last equations are $L = 0$ (not applicable) and $L = 1050$.

10.

A logging company wants to chop down a forest that is 99% pine trees. The Forest Service objects and the logging company, whose CEO is a devious mathematician, suggests that it will only cut pine trees and, when done, the forest will still be 98% pine trees. What percentage of the forest will be chopped down?

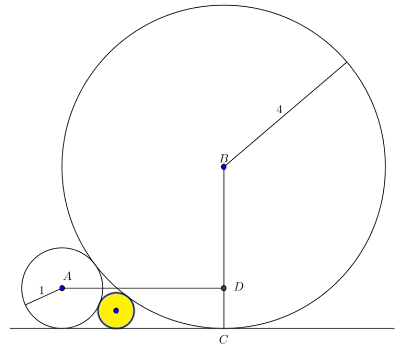
Solution. Suppose the forest has 100 trees; 99 pine trees and one Maple tree. If x pine trees are cut down there will be $100 - x$ trees in the forest; if 98% are pine trees that means that $99 - x = 0.98(100 - x)$. Working on this equation we get $1 = 0.02x$ so $x = 1/0.02 = 50$. Half the forest gets cut. The answer is $\boxed{50\%.}$

11. Three circles are tangent to a horizontal line and to each other, as shown in the figure below. The circle centered at A has a radius of 1; the circle centered at B has a radius of 4. What is the radius of the third yellow circle? The Theorem of Pythagoras could come in handy.



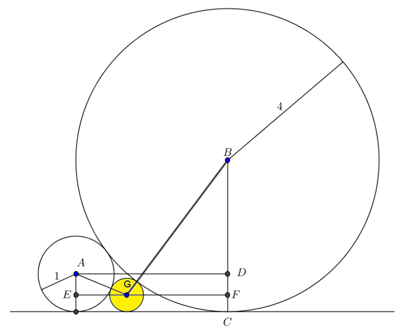
Solution.

Draw the segment BC which is perpendicular to the horizontal common tangent; Draw a line through A perpendicular to BC, intersecting BC at D. We will measure the length of the segment AD in two ways. By Pythagoras $|AD|^2 = |AB|^2 - |BD|^2$. When two circles are tangent, the distance between their centers equals the sum of their radii, so that $|AB| = 1+4 = 5$, while $|BD| = 4-1 = 3$, so that $|AD| = \sqrt{5^2-3^2} = \sqrt{16} = 4$.



For the second measurement we draw the vertical from A to the horizontal tangent; the line parallel to AB through the center of the small yellow circle intersects this vertical at E and intersects BC at F. We let G be the center of the small circle. We clearly have

$$|AD| = |EF| = |EG| + |GF|.$$

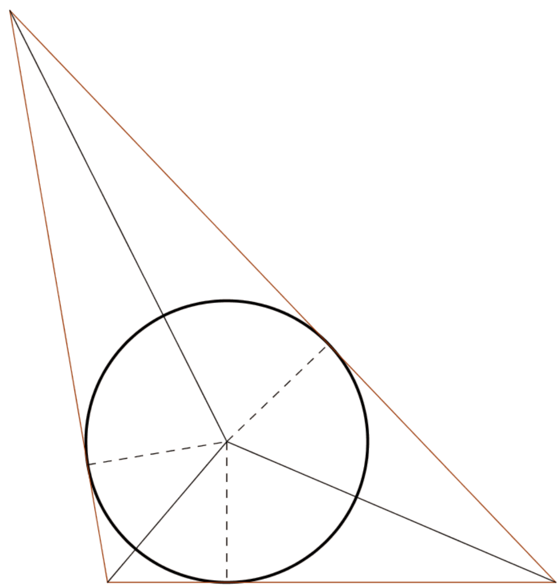


Once again, by Pythagoras, $|EG|^2 = |AG|^2 - |AE|^2$. If we write r for the radius of the small circle, then $|AG| = 1 + r$ and $|AE| = 1 - r$ so that $|EG|^2 = (1 + r)^2 - (1 - r)^2 = 4r$, so $|EG| = 2\sqrt{r}$. Similarly, $|GF|^2 = |GB|^2 - |BF|^2 = (4 + r)^2 - (4 - r)^2 = 16r$ so that $|GF| = 4\sqrt{r}$ and $|AD| = 2\sqrt{r} + 4\sqrt{r} = 6\sqrt{r}$. Equating to the previous measurement of $|AD|$ we get $6\sqrt{r} = 4$ thus $r = (4/6)^2 = (2/3)^2 = 4/9$.

The answer is $\boxed{r = \frac{4}{9}.}$

12. A circle of radius 7 inches is inscribed in a triangle of perimeter P inches and area A square inches. What is the ratio P/A? The answer should be in one of the following forms:

   - An integer.
   - an expression of the form $\frac{a\sqrt{b}}{c}$ where a, b, c are integers.
   - A fraction $\frac{a}{b}$, where a, b are integers with no common divisor other than 1.



Solution. The picture appears here with a few added lines. The solid lines from the center of the circle to the vertices of the triangle divide the triangle into three triangles, each one of which has a side of the original triangle as its basis and a radius of the circle (dashed lines) as its altitude. If the lengths of the sides of the original triangle are a, b, c, then the areas of the three triangles in which it is partitioned work out to

$$\frac{1}{2}ar, \quad \frac{1}{2}br, \quad \frac{1}{2}cr$$

where r is the radius of the circle. Adding up we get the area of th triangle, thus

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)r = \frac{1}{2}Pr.$$

Thus

$$\frac{P}{A} = \frac{2}{r},$$

so in our case $\boxed{\frac{2}{7}.}$