FAU Math Circle

11/7/2015

Math Warm Up

Our warm-up today consists of some questions from Raymond Smullyan's *What's the name of this book* We are in what he calls the island of Baal, populated not only by humans but also by monkeys. These are very evolved monkeys, so they look exactly like humans and also are divided into truthful knights and lying knaves. In the story the visitor to the island is trying to get to a secret place in the island and to get there must pass several tests. We'll only do a few.

- An island inhabitant says "I am either a knave or a monkey." What is he/she?
- The visitor meets inhabitants A and B who say:

  A: At least one of us is a monkey.

  B: At least one of us is a knave.

  What are A, B?
- The visitor then meets inhabitants A and B (not the same as before) who say:

  A: Both of us are monkeys.

  B: Both of us are knaves.

  What are A, B?
- Here is a longer one, pen and paper and writing down possibilities could help. The visitor now has to go through one of four doors, marked X, Y, Z, W. At least one (maybe two, maybe all) lead to the secret place the visitor wants to reach. But if the visitor selects a wrong door, the visitor will be devoured by a fierce dragon. Eight islanders, A, B, C, D, E, F, G, H, each of whom is either a knight or a knave, are standing by and make the following statements.

  A: X is a good door.

  B: At least one of the doors Y or Z is good.

  C: A and B are both knights.

  D: X and Y are both good doors.

  E: X and Z are both good doors.

  F: Either D or E is a knight.

  G: If C is a knight, so is F.

  H: If G and I (that is H) are both knights, then so is A.

  Which door should the visitor choose?

Today's Problems

(10/24/15)
Rules:

- Work on these problems in any order. You will have until about 3:30 for this activity.
- Work alone or in groups.
- **This is NOT an exam. If you have questions, need hints, just ask one of the organizers.** We want to challenge you, not frustrate you.
- Feel free to get up, walk around the room, write on the white boards with the provided markers.
- At 3:30PM, more or less I will ask for solutions, and we will discuss the solutions. Students or groups who have found solutions, time permitting, can present them on one of the white boards.

1. Find the unit digit of 3^{2015}.

2. A grocer mixes two kinds of tea one costs 32 cents a pound and the other costs 40 cents a pound. The grocer plans to sell the mixture at 43 cents a pound, making a profit of 25% on the cost. How many pounds of each kind of tea should be used to make 100 pounds of the mixture? (MMC 2002).

3. At a carnival booth, you are allowed to throw a ball three times at a target. You must hit the target twice in a row to win. You must alternate between throwing with your right and left hand. You may start with whichever hand you wish. You hit the target 6/10 of the time with your right hand and 3/10 of the time with your left hand. What is the probability of winning if you make the best choice of starting hand? (MMC 2003)

4. Several consecutive pages are missing from a book. The sum of the missing page numbers is 332. What is the smallest missing page number? (MMC 2001) A helpful formula could be:

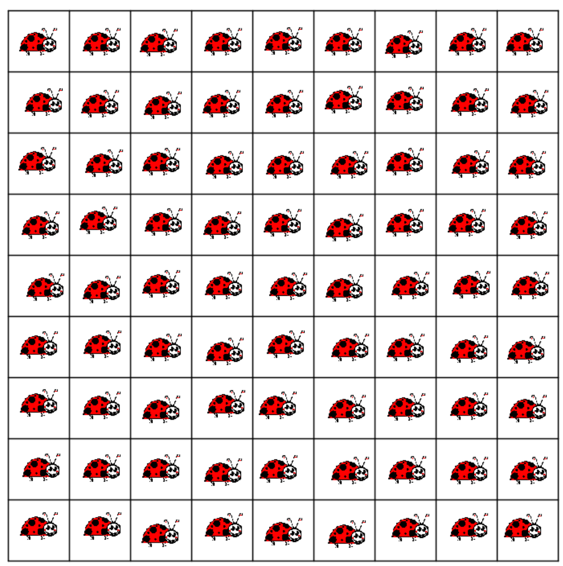
$$1 + 2 + \cdots + m = \frac{m(m+1)}{2};$$

   that is, the sum of the first m positive integers equals m times $m+1$ divided by 2.

5. Find the only five-digit integer x such that $9x$ is obtained by writing down the digits of x in the reverse order. (MMC 2006)

6. A frog can move forward 2 feet or 3 feet on a hop and can also move back 1 foot on a hop. How many sequences of hops are possible that will have moved the frog forward 15 feet after exactly 8 hops? (MMC 2007)

7. Each square of a 9×9 board has a bug sitting on it. On a signal, each bug crawls **along a diagonal** to a square sharing a vertex with the one the bug was on. After this move, some squares end up holding more than one bug, and some squares are empty. Find the smallest possible number of empty squares.(Mos)



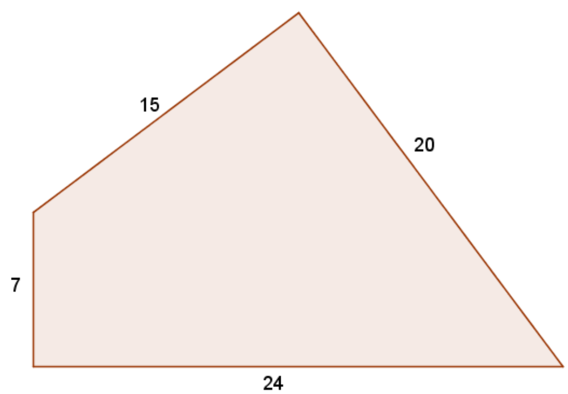
8. In a long division the dividend is 529,565 and the successive remainders from the first to the last are 246, 222, and 542. Find the divisor and the quotient.

9.  Heron of Alexandria said 2000 years ago: To find the area of a triangle of sides of lengths a, b, c find the semiperimeter $s = (a + b + c)/2$. The area is given by the formula

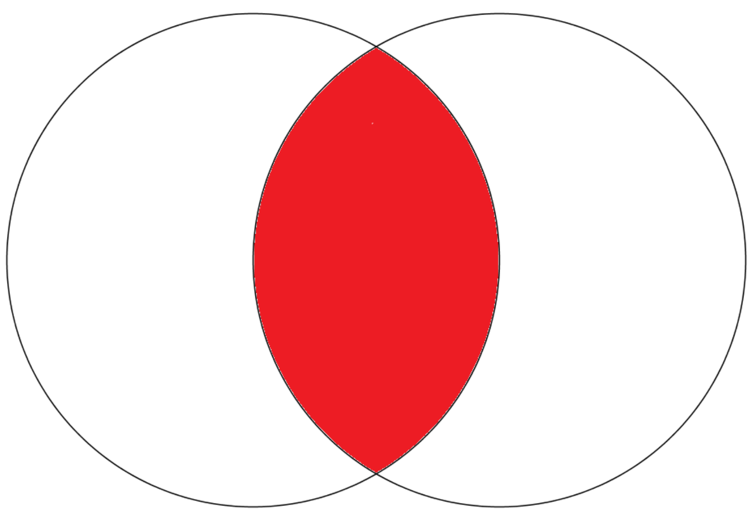
$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

This wonderful formula (and some help from my friend Pythagoras) will help you solve the following problem.

The quadrilateral in the picture below has a base of length 24, another side perpendicular to it of length 7, and the two other sides have length 15 and 20. Find its area. The final expression involves a square root and π.



10. Two circles each of radius 10 are drawn so that the center of each is on the circumference of the other. Find the area of the curvilinear figure common to both circles.

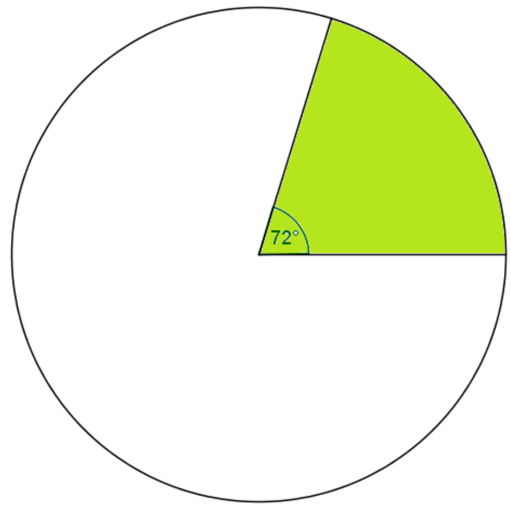


The next page contains some information that could be useful to solve this problem. Can you see where it could be used?

Area of a circular sector

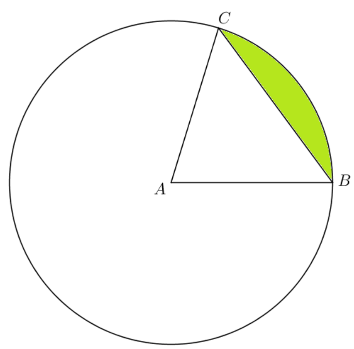
Suppose you have a circle of radius r. A circular sector of central angle a degrees has an area of $\frac{1}{2}r^2\left(\frac{a \times \pi}{180}\right)$. For example, if the circle in the picture on the right has a radius of 3 and the shaded sector has a central angle of 75°, then its area is

$$\frac{1}{2}r^2 \cdot \frac{72\pi}{180} = \frac{r^2\pi}{5} = 1.8\pi$$



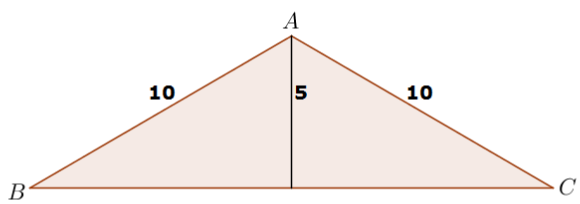
Area of a circular segment

The area of the circular segment pictured on the right is the area of the circular sector minus the area of triangle ABC.



There is something about being isosceles!

The isosceles triangle ABC on the right has two equal sides of length 10 and an altitude of length 5. Find the measure of its angle at A and its area. What does this have to do with the problem at hand?



Sources

MMC: Mathworks Competition.

Mos: A Moscow Math Circle.