FAU Math Circle
10/24/2015

Math Warm Up

- The National Mathematics Salute!!! (Ana)

- Two problems from Raymond Smullyan's *The Gödelian Puzzle Book.*

  - On my next visit to this island I came across three natives A, B and C and was reliably informed that one of the three was a magician. They made the following statements:

    A: B is not both a knave and a magician.

    B: Either A is a knave or I am not a magician.

    C: The magician is a knave.

    Which one is the magician and what type is each?

    Solution. Let us start with any statement, assume first the person making the statement is of one of the types, and see where it leads us.

    Perhaps it is best to start with the simplest statement, which is C's statement; the magician is a knave. If C is a knight, then the statement is true; if C is a knave, it is false. We get only a little bit out of it so far: C cannot be the magician since if C is a knight the magician must be a knave, so either A or B; if a knave the magician must be a knight so again not C. So now we know A or B is the magician. Let us look at B's statement and assume first B is a knave. Then what B says is false; a statement of the form "p or q" is false if and only if p AND q are false; so if B is a knave then A is a knight and B is the magician. With A a knight, A is speaking the truth, so B is not both a knave and a magician, but we just saw B is precisely that. So we have a contradiction, B must be a knight. Then A must B a knave or B is not the magician. Suppose A is a knave. Then A lies, so B is both a knave and a magician; not possible since we saw B has to be a knight. So A is a knight and what A says is true; B cannot be both a knave and a magician. But, more importantly, because B speaks the truth and we saw that A can't be A knave, it must now be true that B is not the magician. So A,B are knights, A is the magician. What about C? Well, C declared the magician to be a knave, so C lied. It follows that C is a knave.

    One gets the same result starting at other points. For example, let us start with A. Suppose A is a knight. Then B is not both a knave and a magician, meaning that B is either a knight or not the magician. Suppose B is a knight. Then since we are assuming at this point that A is a knight, the only way what B says can be true is if B is, in fact, not the magician. No contradiction so far. Now C states that the magician is a knave. That is a very troubling statement because if C is a knave, C is lying and the magician has to be a knight. If C tells the truth, then the magician has to be a knave, but there are no knaves left. So C must be lying, C is a knave, and the magician must be A since B is not the magician and the only knights are A, B.
    So A, B knights, C a knave, A is the magician is logically consistent.

    Is it the only possibility? Assuming still that A is a knight, A's statement would also be true if B is not the magician, but a knave. Then B must lie; since "p or q" is false only if BOTH p AND q are false, we get the contradiction B is not not a magician.

    Let's suppose next that A is a knave, so A is lying. So B must be both a knave and a magician. So B is lying and both A is a knave AND I am not a magician must be false, so A must be a knight, a contradiction.

    We are left with the original conclusion as the only possible one: A a knight and a magician, B a knight, C a knave.

– I then witnessed a trial. A crime had been committed and three suspects, A, B, and C, were being tried. They made the following statements:

  A: I am guilty.

  B: I am the same type as at least one of the others.

  C: We are all of the same type.

  Which one is guilty?

  Solution. A is guilty because A has to be a knight. Assume A is a knave. Now C says we are all of the same type. Since A is a knave this would be a true statement if C is also a knave, but knaves can't tell the truth. So it can't be a true statement, so C must also be a knave and for the statement to be false, B must be a knight. But then B is lying.
  Assuming A is a knave leads to a contradiction, so A must be a knight. Thus A speaks the truth and is guilty. It is not possible to determine what C is. If C is a knave then he, as he should, is lying when C says "we are all the same type." Since then B would be telling the truth, B is a knight. It is also possible for C to be a knight, then all three are knights.

Today’s Problems

(10/24/15)
Rules:

- Work on these problems in any order. You will have until about 3:30 for this activity.
- Work alone or in groups.
- **This is NOT an exam. If you have questions, need hints, just ask one of the organizers.** We want to challenge you, not frustrate you.
- Feel free to get up, walk around the room, write on the white boards with the provided markers.
- At 3:30PM, more or less I will ask for solutions, and we will discuss the solutions. Students or groups who have found solutions, time permitting, can present them on one of the white boards.

1. **(a)** There are three apples on a table. The first weighs 200 grams, the second 300 grams, and the third weighs 400 grams. Gabriela and Sam each take an apple and start to eat at equal rates. Whoever finishes an apple first takes the last one. If each wants to eat as much as possible, which apple should Gabriela take first? **(b)** What if there is a fourth 450 grams apple on the table?

   Solution. **(a)** Gabriela should take the 200 g apple. In this way she will finish first and even if Sam took the 400 g apple, she will be the winner. **(b)** Gabriela should take the 300 g apple this time. If Sam then takes the 200 g apple the best he can do is take the 450 g for his second apple, Gabriela than gets the 400 g apple. She got 700 grams of apple to Sam's 650. If Sam takes the 450 apple at first, Gabriela will finish first and can take the 400 apple as her second apple with the same result as before. If Sam takes the 400 g apple, Gabriela can do even better.

2. Twenty points are placed on a sheet of paper and a line is drawn through every two points. What is the least and what is the greatest number of lines that could be so formed?

   Solution. The least number is 1; if all points are in a line. For the largest number it is possible to place the points in such a way that no three are on the same line; for example by placing them on a circle. So the largest number of lines can be counted as follows. Let's say we number the points from 1 to 20. We can draw 19 lines through point 1; one for each remaining point. We can then draw 18 additional lines through point 2. We can draw 17 additional lines through point 3. And so forth. Once we get to point 19 the only line not yet drawn is one through points 19 and 20. So the total number of lines is

   $$19 + 18 + 17 + \cdots + 2 + 1.$$

   We can break up these 19 numbers into 9 pairs

   $$(1, 19), (2, 18), (3, 17), (4, 16), (5, 15), (6, 14), (7, 13), (8, 12), (9, 11)$$

   adding up to 20, plus the middle number, which is 10. So the sum is $6 \times 20 + 10 = \mathbf{190}$.

3. A chunk of consecutive numbered pages has fallen out of a folder. The first page of the chunk has number 463, and the last has the same digits but in a different order. How many sheets of paper were dropped? (Each sheet is two pages with consecutive numbers.)

   Solution. The last page must have an even number; since it has to be larger than 463 it must be 634. So the total number of pages is

   $$634 - 463 + 1 = 172,$$

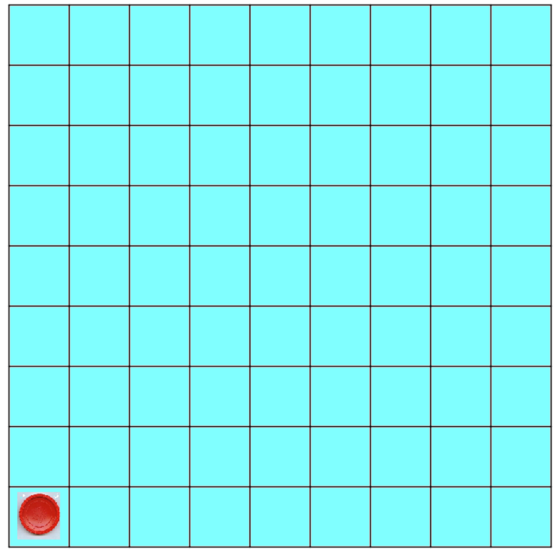
   so the number of sheets is $172/2 = \mathbf{86}$.

4. Mike rips a piece of newspaper into 8 pieces. He rips one of the resulting pieces into 8 pieces. Can he rip the paper into 2015 pieces? If he waits until next year, can he rip it into 2016 pieces? If he waits two years, can he rip it into 2017 pieces? (Assume the original sheet of newspaper is humongous)

   Solution. The key here is to notice that at every stage the number of pieces of paper minus 1 is a multiple of 7. It is so at the beginning, since 8-1=7. Every time a piece of paper now gets ripped we replace one fragment by 8, so we increase the number of pieces by 7. So if we had x pieces and $x - 1$ was a multiple of 7, then after the next rip we have $x + 7$ and $(x + 7 - 1) = (x - 1) + 7$ is again a multiple of 7. We see that by ripping in this way we can end with a number of pieces equalling any multiple of 7 plus 1, and only such numbers. Now $2015 - 1 = 2014$, $2016 - 1 = 2015$ are not multiples of 7, but $2017 - 1 = 2016 = 7 \times 288$ is a multiple of 7. The paper can be ripped into 2017 pieces, but not into 2015 or 2016 pieces.

5.

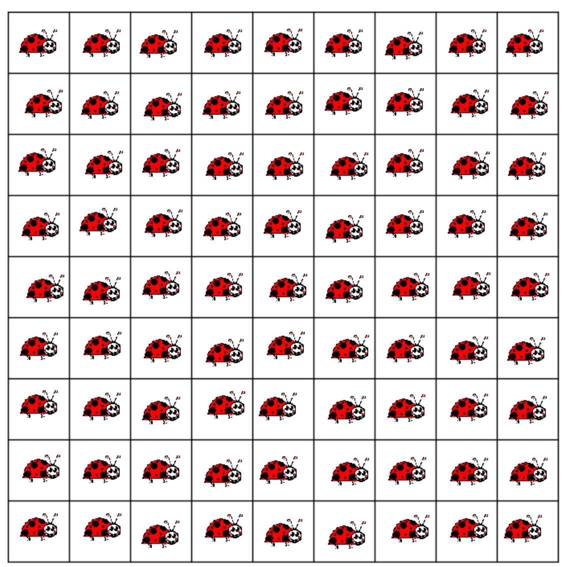
A marker has been placed in the lower left corner of a 9×9 board. Two players take turns choosing a direction, right or up, and the number of spaces to move in that direction. The winner is the last one to make a move. Which of the players, first or second to move, can always win, and what is the strategy to do so.



Solution. The second player can win. The moment a player moves the marker to a square in the right or upper border, the next player to play wins; this player can then move the marker to the top right corner and the game is over. The first player has to move the marker to a position in the lower row or in the left row of squares. From then on the second player should always push the marker to the next available diagonal position, either to the right or above the position where the first player leaves the marker. In this way the first player can never get a into a square in the diagonal, while the first player puts the marker in diagonal positions that are closer and closer to the top right corner.. Eventually he will be able to place the marker in the square whose upper right vertex coincides with the lower left vertex of the top right square. At this point the first player has no choice but moving into the top or right row, and the second player wins at the next move.

6.

Each square of a 9×9 board has a bug sitting on it. On a signal, each bug crawls onto one of the squares that shares a side with the one the bug was on. **(a)** Prove that one of the squares is now empty (that is, there is at least one square now with two or more bugs on it). **(b)** Can the bugs move so that there will be exactly one empty square?



Solution. **(a)** Number the squares consecutively from left to right top to bottom with the numbers from 1 to 81. Bugs in an even numbered square will move into an odd numbered square, bugs in an odd numbered square will move into an even numbered square. However, there are 40 even numbered squares and 41 odd numbered squares, so one of the odd numbered squares will have to remain empty after the move. **(b)** Yes.

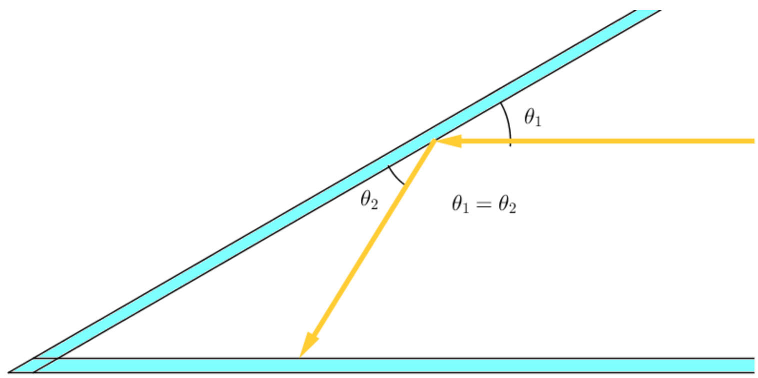
Rows 1 and 2, 3 and 4, 5 and 6, 7 and 8 can move in a cycle: Take, for example, rows 1 and 2. The first 8 bugs of row 1 can crawl one square to the right, the last bug of row 1 crawls one square down, the bugs in the last 8 squares of row 2 crawl one square to the left, and the bug in the first square of row 2 moves one square up. In this way there are no empty spaces in rows 1 and 2. The same movement can be repeated in rows 3 and 4, 5 and 6, 7 and 8. In row 9, the bugs in the first 8 squares move one square to the right, the bug in the last square of row 9 moves one square to the right. The only empty square is the first square of row 9.

7. Sam and Alex live in the same apartment building and leave for school at the same time. Each of Sam's steps is 10% longer than Alex's, but Sam takes 10% fewer steps per minute than Alex. Who will get to school first?

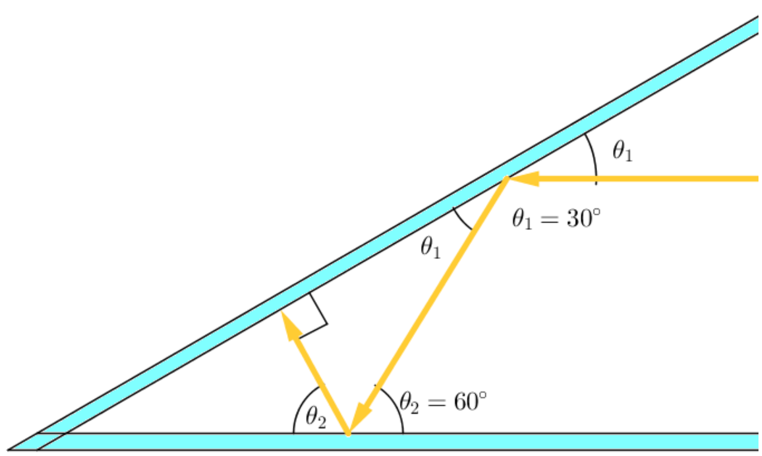
   Solution. The answer is Alex. Suppose for example that Alex can make 10 steps per minute, each one 1 foot long. Then Alex is moving at the rate of 10 feet per minute. Now Sam's steps are 10% longer than Alex's steps, so Sam's steps are 1.1 feet long. But in a minute he takes 10% fewer steps, thus 9 steps. This means that Sam only covers $9 \times 1.1 = 9.9$ feet in a minute. Alex is faster and will arrive first at school.

8. **(a)** Two mirrors form a 30° angle. A light beam enters this angle parallel to one of the sides and is reflected from the sides according to the usual law that the angle of incidence is equal to the angle of reflection. Prove that the beam will eventually leave the angle. How many times will it reflect off the mirrors before leaving? **(b)** What if the angle between the mirrors is 20°? **(c)** What if it is 50°?

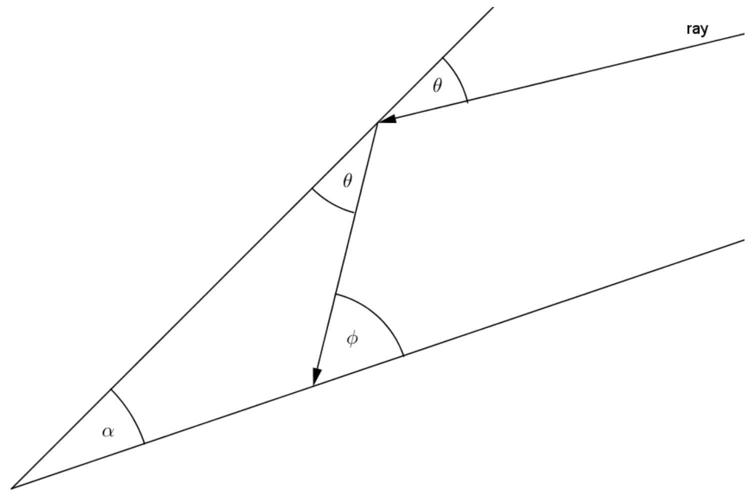
   The picture shows a light ray entering the angle and the first reflection.



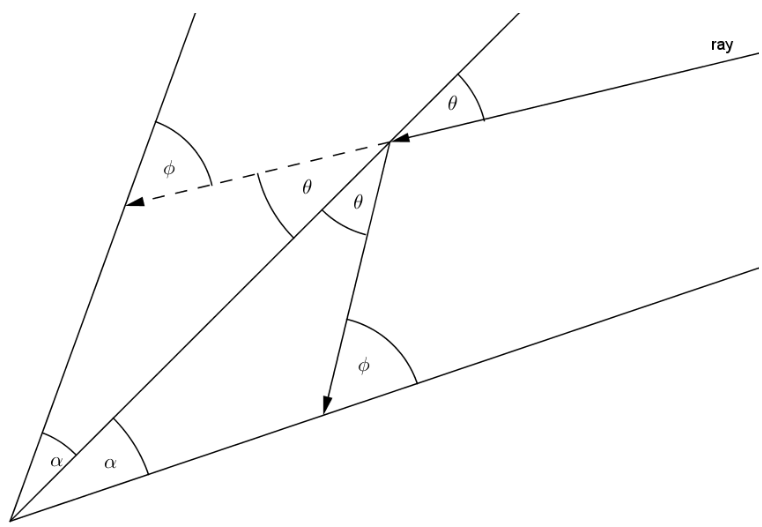
   Solution. **(a)** At the first reflection the angle of incidence (actually, the angle complementary to the angle of incidence) is 3o degrees, so it reflects with the same angle. A bit of geometry shows that it hits the lower mirror at an angle of 60 degrees, reflecting at 60 degrees. As it hits the upper mirror we have a triangle with one 30 degree angle, one 60 degree angle, so the third angle, at which the ray hits the upper mirror, must be 90 degrees. At this point the ray goes back following in reverse the same path it came in. As it leaves it will have reflected 5 times.



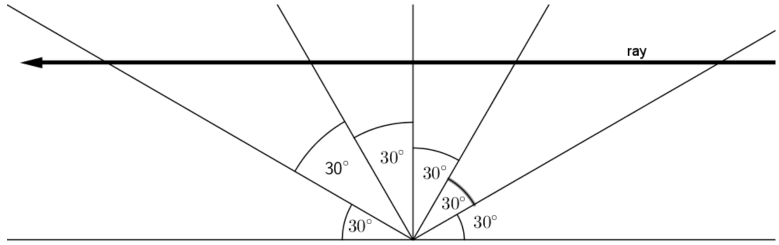
But this method can get sort of difficult for smaller angles. Here is a trick making it all easy. Suppose we have two mirrors forming an angle we will call α. A light ray hits one of the mirrors forming an angle θ as shown; it gets reflected forming an angle θ, hits the second mirror forming an angle ϕ. Let us call the mirror hit at first by this ray the first mirror.



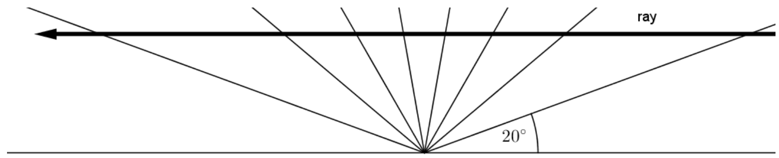
If we reflect the second mirror about the first; that is, place it on the other side of the first mirror forming the same angle α and we think of the ray going straight through the first mirror, we get the following picture,in which the dashed arrow represents the light ray continuing in a straight line.



We see that the angle of incidence with respect to the reflected second mirror is the same as it would have with that mirror in its original position. We can thus just reflect the mirrors about each other, keep the light ray straight, until it has no more mirrors to hit. For the question in part (a) the picture then looks as follows.

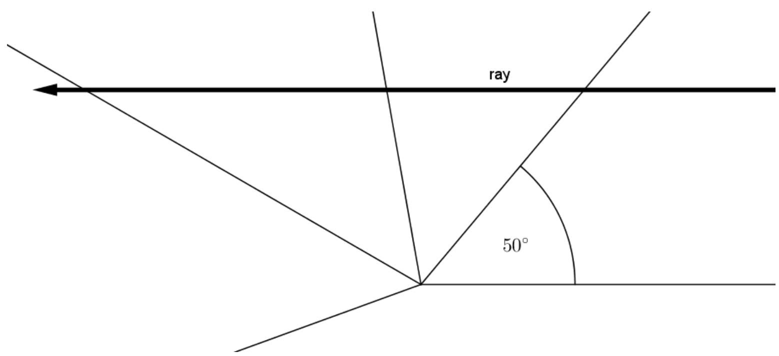


The light ray is out after 5 reflections. For part (b), the picture looks like:



The ray is out after 8 reflections.

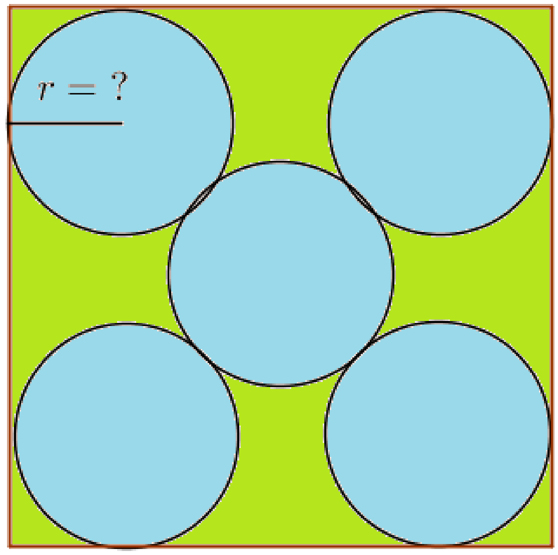
For part (c) we have the following picture



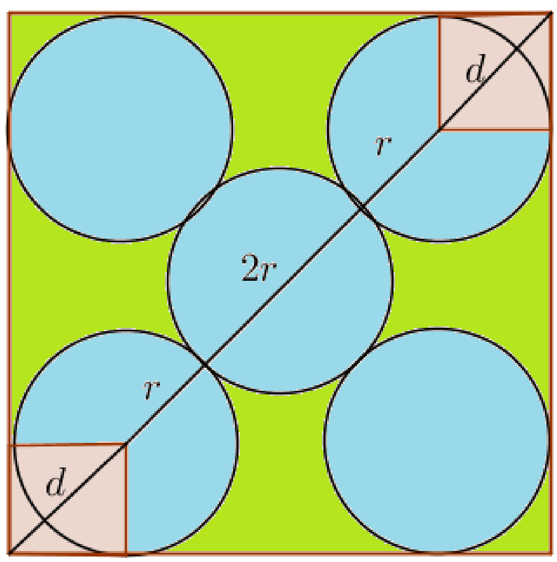
The ray is out after 3 reflections.

9.

The picture on the right shows a square and five circles of the same size inside the square. The four corner circles are tangent to the middle circle and to the sides of the square. If the side of the square has length L, what is the length of the common radius of the circles? (Spoiler alert: A correct answer involves a square root).



Solution. The Theorem of Pythagoras tells us that the diagonal of a square of side length L has length $\sqrt{2}\,L$. For the square in question we can also measure its diagonal in a second way. The diagonal equals $2d+3r$, see the picture:



Since d is the diagonal of a square of side r, we have $d = 2\sqrt{r}$. Equating the two expressions for the diagonal of the large square we get

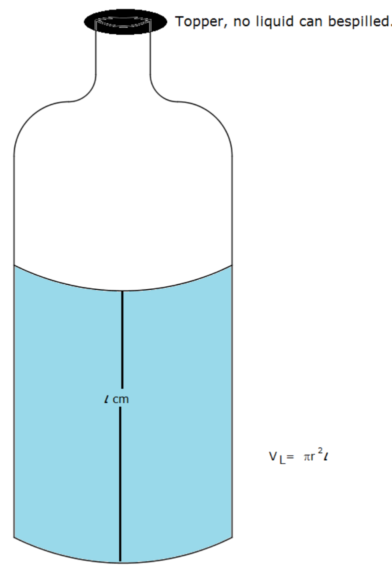
$$\sqrt{2}L = (4 + 2\sqrt{2})r$$

so that

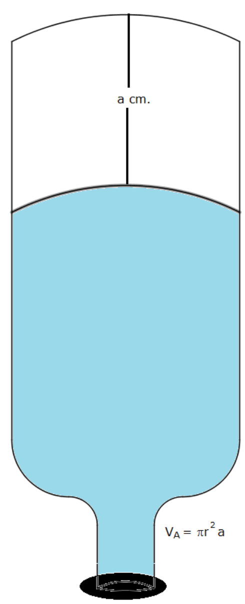
$$r = \frac{\sqrt{2}\,L}{4 + 2\sqrt{2}}.$$

10.

A bottle with circular bottom is partially filled with liquid. There is a stopper at the top so no liquid can be spilled. You want to measure its volume but you only have a ruler. How can you measure its volume? It helps to know that the volume of a circular cylinder having a base of radius r and height h is $V = \pi h r^2$. Suppose the center of the base of the bottle is marked, so measuring the radius is easy.



Solution. The volume of the bottle is $V = V_L + V_A$, where V is the volume of the liquid and V_A the volume of air in the bottle. Using a ruler you can measure the height of the liquid, so you get V_L using the formula for area of a cylinder. Now turn the cone over,now the air is in the cylindrical part so you can easily compute the volume of the air in the bottle.



11. Solve the following cryptic multiplication. Each asterisk stands for a digit and each digit is a prime (2, 3, 5, or 7).

$$
\begin{array}{ccccc}
 & & * & * & * \\
 & & \times & * & * \\
\hline
 & * & * & * & * \\
* & * & * & * & \\
\hline
* & * & * & * & *
\end{array}
$$

Is there more than one solution?

Solution. The answer is

$$\begin{array}{ccccc} & & 7 & 7 & 5 \\ & & \times & 3 & 3 \\ \hline & 2 & 3 & 2 & 5 \\ 2 & 3 & 2 & 5 & \\ \hline 2 & 5 & 5 & 7 & 5 \end{array}$$

. There is no other solution.