FAU Math Circle
10/3/2015

Math Warm Up

- The National Mathematics Salute!!! (Ana)

- What is the correct way of saying it: 5 and 6 are 12 or 5 and 6 is 12?

  Solution. 11 and 5 are 6 are 11.

- For the next three questions we turn to the island of Knight and Knaves. This island, located in the eastern sector of the Bermuda triangle is populated by three groups of people; the knights, the knaves, and the normals. The knights always tell the truth; they cannot lie even to save their lives. The knaves always lie; every statement they make is false. The normals tell the truth sometimes, other times they lie.

  Many years ago[1], during the reign of queen Drumelia 1, a law was passed that knights could only marry knaves, and knaves knights, while normals could only marry normals. Thus given any married couple one of them is a knave and the other one a knight, or both are normal.

  – We first consider a married couple, Mr. and Mrs. Rumpelstiltskin, also known as Mr. and Mrs. R. They make the following statements.

    Mr. R: "My wife is not normal."

    Mrs. R: "My husband is not normal."

    What are Mr. and Mrs. R?

    Solution. Neither Mr. nor Mrs. R. can be knaves; they would be telling the truth. But that means that they can't be knights. So they are normals (and lying)

  – Suppose instead that they had said:

    Mr. R: "My wife is normal."

    Mrs. R: "My husband is normal."

    What are Mr. and Mrs. R in this case?

    Solution. Same as before. Neither can be a knights, thus neither can be a knave.

  – We now consider two married couples; Mr. and Mrs. Munchausen (also known as Mr. and Mrs. M) and Mr. and Mrs. Struwelpeter (Mr. and Mrs. S). Here are their statements.

    Mr. M: "Mr. S. is a knight."

    Mrs. M: "My husband is right; Mr. S. is a knight."

    Mrs. S: "That's right, my husband is a knight."

    What are each of the four people and which of the three statements are true.

    Solution. Mrs. S cannot be a knight; if so, her husband would be a knave and she has just lied about her husband something knights can't do. But she can't be a knave; then she would be telling the truth about her husband. So Mrs. S, thus also Mr. S, are normal. But then both Mr. and Mrs. M are lying; since in any non normal couple at least one has to be a truthful knight; we see they are also normal. All four people are normal, all statements are false.

[1] From R. Smulyan's *What is the Name of This Book*

Today's Problems

(10/3/15)

1. 201 identically shaped gears are in a row. Each gear, except the first and the last, is connected to its left and right neighbor.

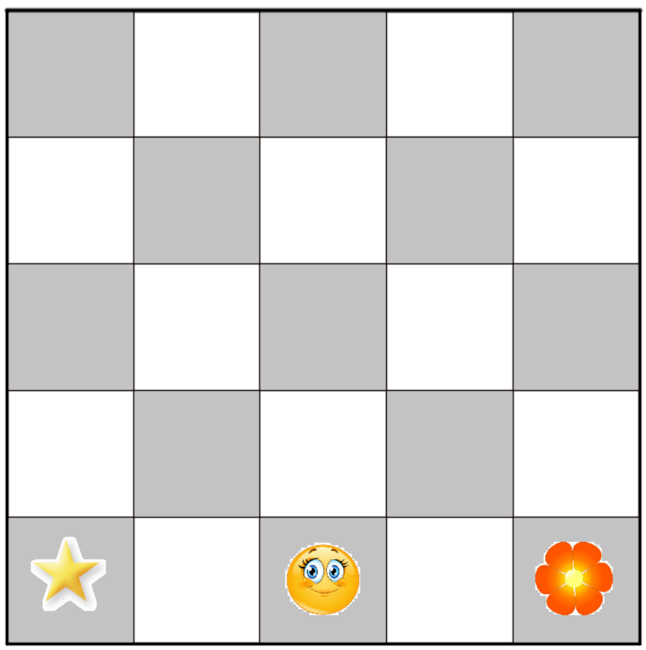
   If the leftmost gear is turned clockwise, is the rightmost gear going to rotate clock or counterclockwise?

   Solution. 201 is an odd number. The answer is clockwise.

2. Suppose that the 201 gears are placed in a circle in such a way that every gear is connected to two neighbors. One of the gears is turned clockwise. What happens then?

   Solution. Nothing. In fact, turning any gear, clockwise or counterclockwise is impossible.

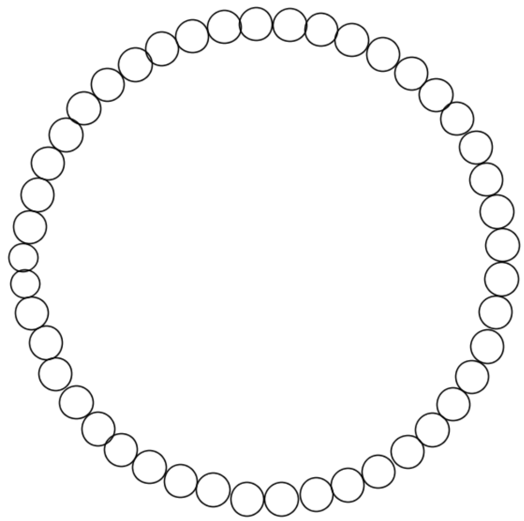
3. A checker is placed on the bottom left corner of a 5×5 checkerboard. The checker is supposed to tour the board, visiting each square exactly once. But, each time it moves, it can only move one square to the left, or one square to the right, or one square up, or one square down.



   (a) Can the checker end at the square marked with the flower? (For each question, either indicate a tour or explain why there is none).
   (b) Can the checker end at the square marked with the smiley face?
   (c) Can the checker end back at the square with the star (so visiting that square twice, in a way)?

   Solution. There are many routes from the star to the flower and to the smiley. There is none taking the checker back to the star. Drawing all possible routes and seeing that none leads back to the star can take a very long time, so we need to do something different. The board has a total of $5 \times 5 = 25$ squares. The checker starts in one of these squares, it has 24 more squares to visit. Since it visits one new square with each move, the checker has to move 24 times to visit all squares of the board. It starts on a gray square, after every second move it is again on a gray square, so after 24 moves it must be on a gray square, separated by at least one square from the gray square with the star. The answer to part (c) is NO.

4. Fortyfive (45) little circles are placed in a circle, as shown below.



They are to be painted using two colors, some circles are to be painted blue, others red.

(a) Prove that there must be two adjacent circles having the same color (adjacent=next to each other).

(b) Prove that somewhere we must have two circles of the same color separated by exactly two circles.

Solution. Suppose no two adjacent circles have the same color. That means they alternate in color. Starting at a red circle go around the big circle; the pattern will be red, blue, red, blue, etc; every second circle is blue. The second, fourth, sixth, and so forth circle we visit will be blue. Because 44 is even, the 44th circle visited will be blue making the final circle, the one adjacent to the first one, red. This shows that supposing we can alternate the colors leads to an impossibility, so the colors cannot alternate. This is part (a).

Part (b) is quite similar, suppose no two circles are separated by two circles of the same color. Suppose we have two red circles separated by a number of blue circles. How many blue circles can there be in the group? The answer is one or three. If there are two we have a contradiction with what we assume, but also if there are four or more; then we have two blue circles separated by exactly two blue circles. Of any group of three blue circles sparating red circles, remove 2. Now we have the red circles separated by at most one blue circle, and while the number of circles has diminished and changed, **it is still an odd number of circles.** Now consider blue circles separated by red circles. As before, there can only be one or three red circles between any two blue circles. If there are three remove two. The end result is that we still have an odd number of little circles left, but now alternating in colors. An impossible situation has been reached.

5. (a) How does the area of a rectangle change if one of its sides increases by 10% while the other decreases by 10%? (b) Same question for the perimeter.

Solution. If the sides of the rectangle are denoted by a and b, the area is $a \times b$ if we increase a by 10% it becomes $a + (a/10) = 11a/10$; if we decrease b by 10% it becomes $b - (b/10) = 9b/10$. The area is now

$$\frac{11a}{10} \times \frac{9b}{10} = \frac{99ab}{100}.$$

Then

$$\text{new area}/mboxoldarea = \frac{\frac{99ab}{100}}{ab} = \frac{99}{100} = 0.99$$

so the new area is 99% of the old one. The answer is: The area decreases by 1%.

For the perimeter, the old perimeter is $2a + 2b$, the new one is

$$2 \times \frac{11a}{10} + 2 \times \frac{9b}{10} = \frac{22a + 18b}{10}.$$

There is not as easy an answer as for the area. For example if the increasing side a is larger than b, then one can see the perimeter gets larger, the percentage increase depending on how much larger a is than b. If b is larger than a, the perimeter decreases. If $a = b$, the perimeter stays unchanged.

6. A ship is on its way from one of the Caribbean islands to Miami. Halfway through the trip the ship gets a hurricane warning so it doubles its speed and arrives at Miami 3 hours ahead of schedule. How long was the whole trip.

   Solution. There are many ways of doing this problem, but here is a useful one. If an object (a runner, a rabbit, a boat, an airplane, a zombie) covers half a distance at speed v and the other half at speed w, the average speed for the whole distance is NOT the arithmetic mean $(v + w)/2$. It couldn't be, otherwise if the object does the first half at 10 mph and then sits down so that it never completes the distance, the average speed is not 5 mph but infinity miles per hour. The average speed is the *harmonic mean*, which is

$$\frac{2vw}{v + w}$$

   In our case the ship goes at one speed, let's call it v, its normal speed, for half the journey, $2v$ for the other half, so its average speed is $\frac{2(2v)v}{2v + v} = \frac{4v^2}{3v} = \frac{4}{3}v$. If it goes at 4/3 the normal speed it should arrive in $1/(4/3) = 3/4$ the usual time, so it shaves off one fourth of the usual time. So 3 hours is one fourth of the usual time, so normally the ship takes 12 hours for the voyage. This trip lasted thus 9 hours.

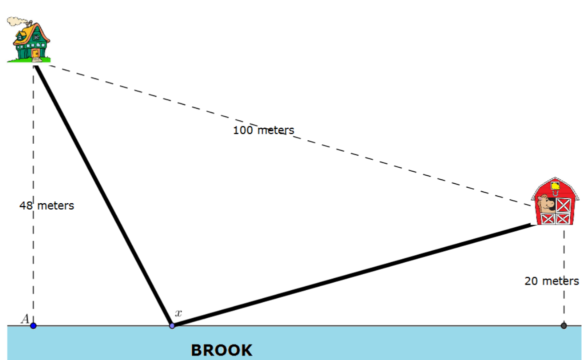
7. A passenger fell asleep on a train when the train was halfway to his destination. When he woke up, he noticed that he had half as far to go as he went during the time he slept. For how much of the journey did he sleep?

   Solution. He slept for 2/3 of 1/2 of the journey, so for one third of the whole journey.

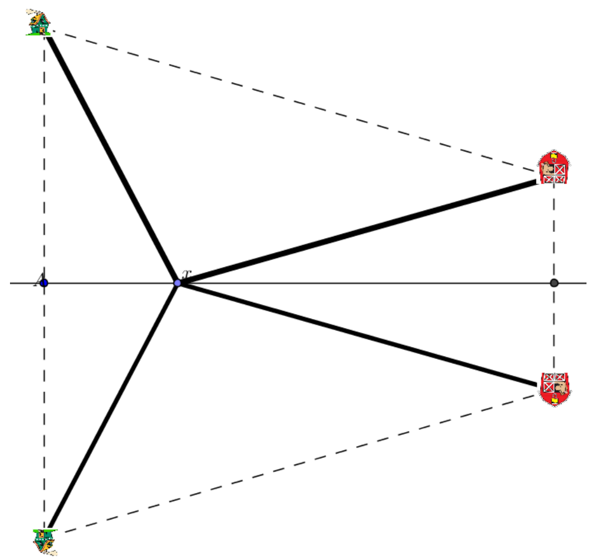
8. A notepad costs a whole number of cents. If 10 notepads cost more than \$11 and 9 notepads cost less than \$10, how much does a notepad cost.

   Solution. If 9 notepads cost less than $10 \times 100 = 1000$ cents that tells you something about the highest price of a notepad; 9 goes into 1000 a total of 111 times, with a remainder of 1. So 111 cents is the highest possible price for the notepads. We now see that $10 \times 111 = 1110 > 1100$, so 111 cents is quite possible. If we reduce the price by one cent, we see that $10 \times 110 = 1100$ exactly 1100 cents or 11 dollars, which is not greater than \$11. The answer is 111 cents or \$1.11.

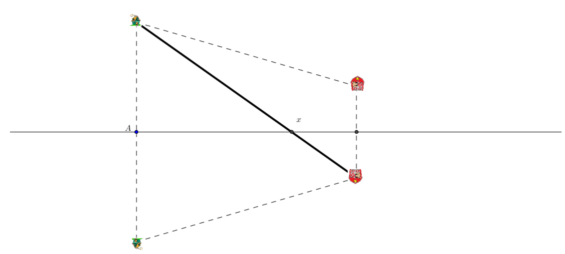
9. Every day Jack, who lives in a house 48 meters away from a brook has to go from his house to the brook to fetch a pail of water and bring it to his horse, which he keeps in a barn 100 meters away from his house. The barn is 20 meters away from the brook. Jack wants to go in a straight line to the brook, and from there straight to the barn, following a path like the one shown in the picture. In the picture x marks the spot where Jack gets to the brook. How far away from A should x be so that Jack has to walk as little as possible?



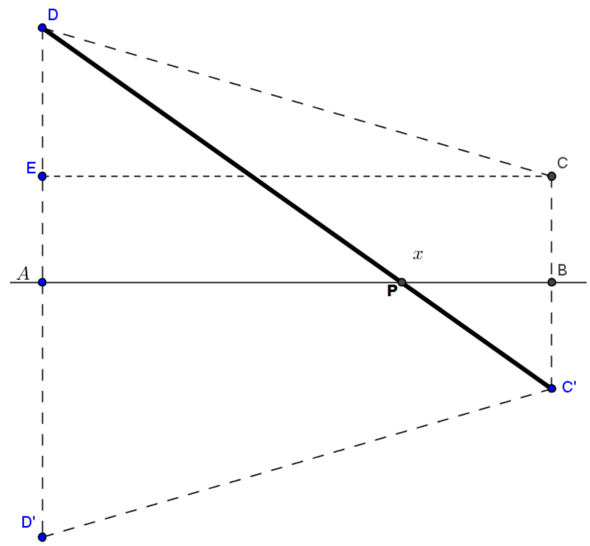
Solution. Suppose we reflect the picture with respect to the brook line, ignoring the water, so now it looks like:



It should be clear that the distance from x to real barn is the same as that from x to reflected barn so that we need to find the point x such that the distance from the house to the reflected barn is least possible. The shortest distance between two points is given by the straight line segment joining them, so we need to select x so house, x, reflected barn ar in a line:



To figure out the location of x it might be good to have a diagram without the pictures of house and barn, but with a few mor points, and one extra segment, drawn.



In this picture $|AP| = x$. It will be convenient to know the length $|AB|$ of AB. We see that $|AB| = |CE|$ and by the Theorem of Pythagoras,

$$|CE| = \sqrt{|CD|^2 - |DE|^2};$$

we see that $|DE| = |AD| - |AE| = |AD| - |BC| = 48 - 20 = 28$ meters. It is given that $|CD| = 100$ meters so that

$$|AB| = |CE| = \sqrt{100^2 - 28^2} = \sqrt{9216} = 96.$$

So AB is 96 meters long.

Triangles ADP and $BC'P$ are similar. In fact, both are right triangles and $\angle APD = \angle BPC'$. That means that corresponding sides are proportional so that we get

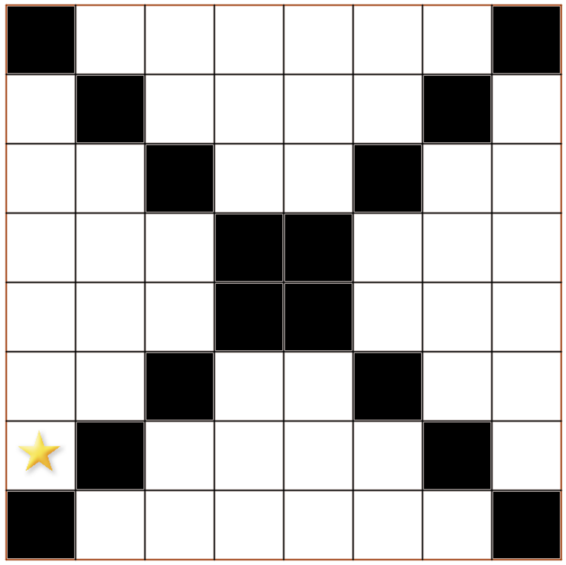
$$\frac{|AP|}{|AB| - |AP|} = \frac{|AD|}{|BC'|};$$

using $|AP| = x$, $|AB| = 96$, $|BC'| = |BC| = 20$, we get

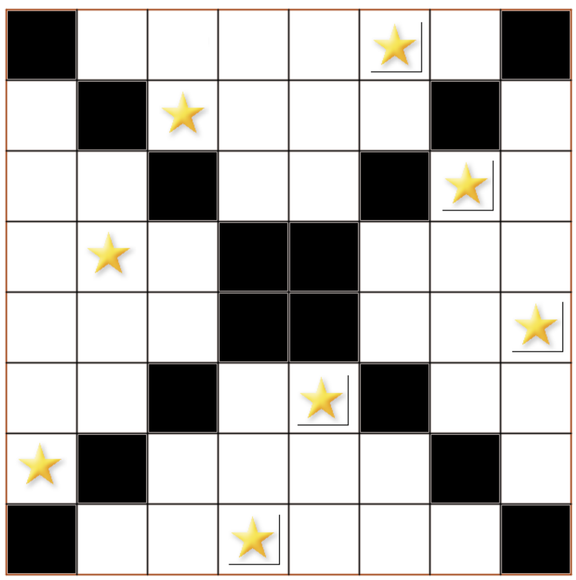
$$\frac{x}{96 - x} = \frac{48}{28}.$$

This can be solved to get $x = 1152/17$ approximately $67.t6$ meters.

10. A star has been placed in one of the white squares of the board, as shown. Place (or draw) 7 more stars in another 7 white squares, so that no two star are in line horizontally, vertically or diagonally.



The only solution is



Trial and error can work.

11. The midpoints of the sides of a triangle have been marked. Then the triangle is erased leaving only the marked points? How can the triangle be recreated using only a compass and a straightedge (and a pencil)?

    Notice: Using only a pencil, a compass, and a straightedge means that you are ONLY allowed to do the following:

    - You may draw a line through any two points (or join them by a segment).
    - You may use any two points to draw a circle as follows: You can use one point as center and use the distance between the points as the radius; that is, you can put the pointy pinchy part of the compass on one point, open until the pencil of the compass is at the other point, and then you can draw the circle.
    - If two lines intersect, or if a line intersects a circle, or if two circles intersect, you may mark the intersection points and use them to draw more lines and/or circles.

Solution. A preliminary construction one needs to know how to do is: Given a line and a point not on it, draw a line parallel to the given line through the given point; do this using only a compass and straightedge. Last session we discussed how to draw perpendicular lines, and that is actually needed to draw parallels.

Suppose we are given a line ℓ and a point P not on it. Assume two points A, B are marked on the line.

- Place a compass at P with opening PA.
- Draw the circle with center P, radius PA. That circle intersects the line ℓ at A (of course) and at another point C.
- Draw the perpendicular bisector of AC:
  (a) With center at A, opening AC draw a circle.
  (b) With center at C, opening AC draw a circle.
  (c) The two last circles intersect at points R, S.
  (d) the line through R, S is the perpendicular bisector of AC. Call it m.
- The point P is on m; the line m is the line through P perpendicular to ℓ. It is possible that $P = R$ or $P = s$, but $R \neq S$. I will assume $P \neq R$; if it is, replace R by S in the construction to follow. Draw a circle with center at P, opening PR. It intersects the line M again at a point T.
- Draw the perpendicular bisector of RT. That is the line through P parallel to ℓ. Do you know why?

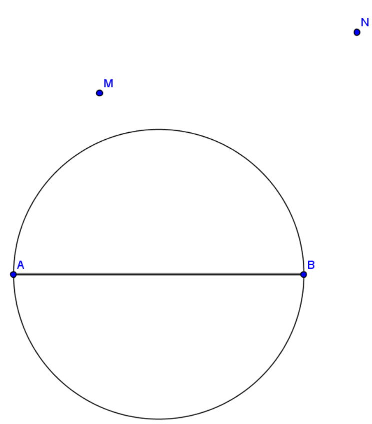
For our construction we will use the following basic fact:

The segment joining the midpoints of two sides of a triangle is parallel to the third side. So if the remaining midpoints are called M, N, P, the line trough P parallel to the line MN must contain one of the sides of the triangle; call it ℓ. The line through M parallel to NP, call it m, also contains one of the sides of the triangle. Finally. so does the line n through N parallel to MP. The vertices of the triangle are given by the intersection of these three lines.

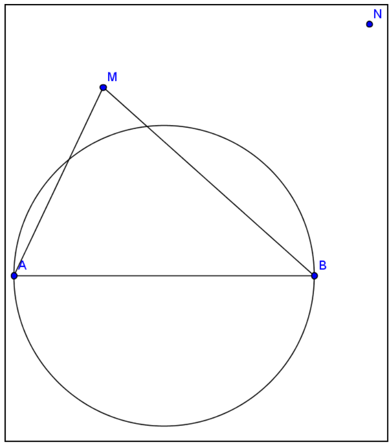
12. Suppose we have points M and N and a circle of diameter AB, positioned in the plane as in the diagram. How can you use only a pencil and a straightedge to drop a perpendicular from M to AB and from N to the continuation of AB?

    Notice: Using only a pencil and a straightedge means that you are ONLY allowed to do the following:

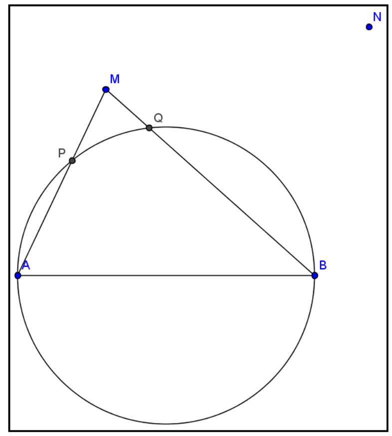
    - You may draw a line through any two points (or join them by a segment).
    - If two lines intersect, or if a line intersects the circle, you may mark the intersection points and use them to draw more lines.



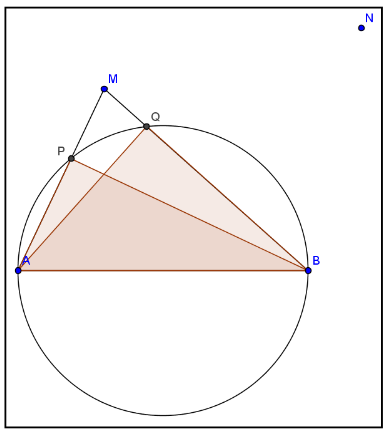
Solution.



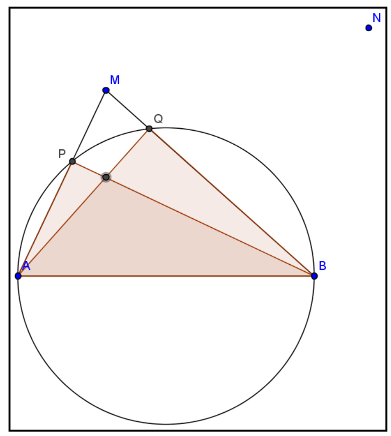
Draw lines from M to A and to B



Mark the points P, Q of intersection with the circle.

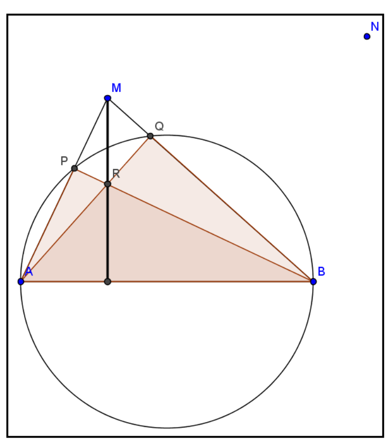


Draw the triangles APB and AQB

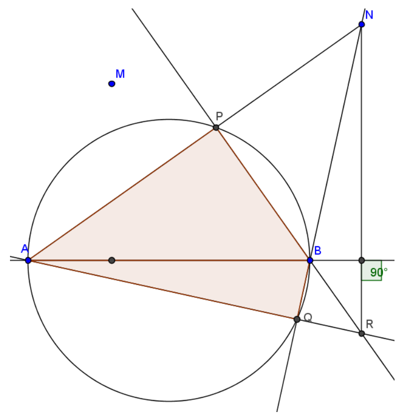


Mark the point R where segments AQ, BP intersect.

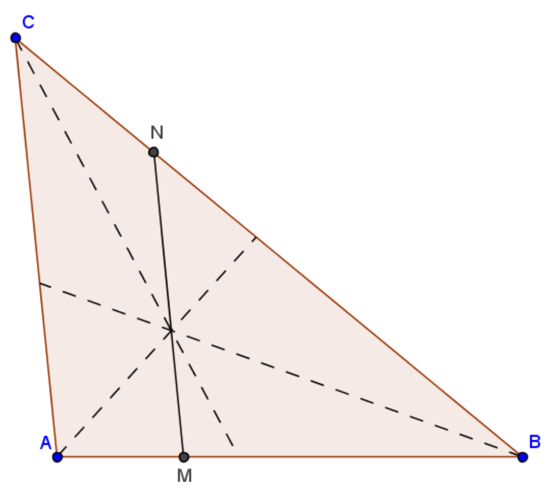
Because angles APB and AQB are inscribed angles subtending the diameter, they are right angles, meaning that AQ and BP are altitudes of triangle AMB. The three altitudes of a triangle intersect at a point; that point has to be R since two altitudes already intersect there. Thus the line from M through R to AB must be an altitude, hence perpendicular to AB.



The solution for the point N is similar, except that now the segments AQ, BP have to be continued outside of the circle to get the intersection R. The point R is again the intersection of two of the altitudes of triangle ANB, hence so is the line through N and R, which will be perpendicular to AB. The final diagram is below.

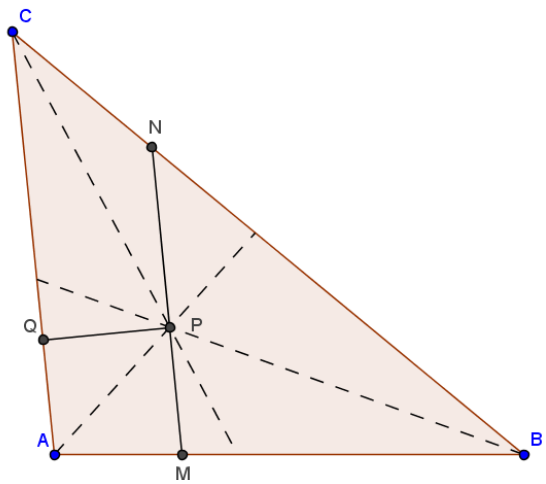


13. In triangle ABC a line was drawn through the intersection of all three angle bisectors, parallel to side AC. This line intersects sides AB and BC in M and N respectively.



Prove that $|AM| + |CN| = |MN|$.

Solution. From the intersection of the bisectors draw a perpendicular line to side AC. Since AC and MN are parallel, it is also perpendicular to MN.



This line intersects AC at a point Q. Since triangle QCP is a right triangle, $\angle PCQ = 90° - \angle CPQ$. But because line CP bisects the angle at C, we see that we also have $\angle PCN = 90° - \angle CPQ$. Since segment PQ and MN are perpendicular, $\angle CPN = 90° - \angle CPQ$; that is $\angle CPN = \angle PCN$ implying that triangle CNP is isosceles and $|CN| = |PN|$. Similarly one sees that $|AM| = |MP|$. Thus $|AM|+|CN| = |MP|+|PN| = |MN|$.