FAU Math Circle

5/09/2015

Math Warm Up

1. I will need three volunteers for this. I will guess what the sum is of five numbers by knowing only the first number. Sort of. You will tell me how it works.

Today's Problems

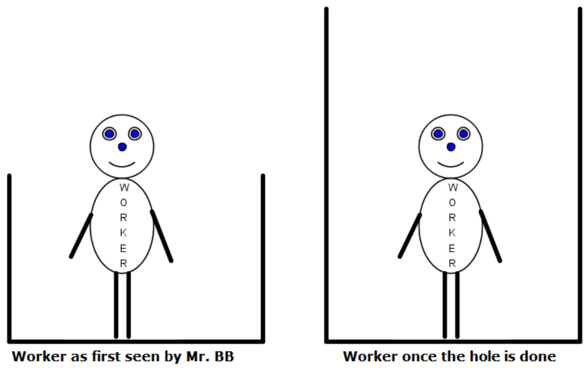
Rules:

- Work on these problems in any order. You will have until about 3:30 for this activity.
- Work alone or in groups.
- Feel free to get up, walk around the room, write on the white boards with the provided markers.
- At 3:30PM, more or less I will ask for solutions, and we will discuss the solutions. Students or groups who have found solutions, time permitting, can present them on one of the white boards.

1. Homer began peeling a pile of 44 potatoes at the rate of 3 potatoes per minute. Four minutes later Christen joined him and peeled at the rate of 5 potatoes per minute. When they finished, how many potatoes had Christen peeled?

   Solution. In the first four minutes, peeling alone, Homer peeled 12 potatoes, so 32 potatoes remained as Christen joined the fun. Suppose they are done in t minutes. Then Homer peeled $3t$ potatoes, Christen peeled $5t$ potatoes and $3t + 5t = 32$; that is, $8t = 32$ so $t = 4$. The number of potatoes peeled by Christen is thus $5t = 5 \times 4 = 20$.

2. On his morning stroll, Mr Busybody encountered a laborer digging a hole. "How deep will the hole be?" he asked. "Guess," replied the workingman, who stood in the hole. "My height is exactly 5 feet and 10 inches and when I'm done digging this hole will be twice as deep as it is now and my head will be **twice** as far below ground as it is now above ground."



How deep will the hole be?

Solution. Let h be the height of the worker, so $h = 5'10'' = 70''$. If originally the hole is x deep, then the worker is $h - x$ above the hole. The final depth of the hole is thus

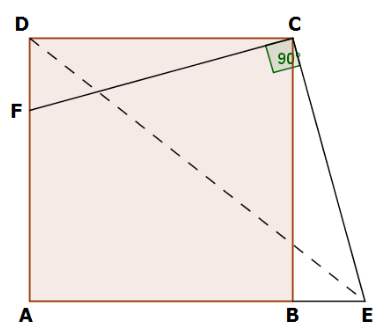
$$2x = h + 2(h - x) = 3h - 2x,$$

so that $4x = 3h$ or $2x = 3h/2$. The depth of the hole will thus be $\frac{3}{2} \times 70 = 105$ inches or 8 feet, 9 inches.

3. If 6 cats can eat 6 rats in 6 minutes, how many cats will it take to eat 100 rats in 100 minutes?

Solution. To say that 6 cats can eat 6 rats in 6 minutes is the same as saying that each cat takes 6 minutes to eat one rat. In 100 minutes each cat can eat 100/6 rats, so 6 cats can eat $6 \times \frac{100}{6} = 100$ rats. The answer is $\boxed{\text{6 cats.}}$

4. In the picture below, $ABCD$ is a square. A point F is marked off on side AD. At C a perpendicular is drawn to CF meeting AB extended at E. If the area of the square $ABCD$ is 256 square inches and the area of the triangle CEF is 200 square inches, how long is the segment BE?



Solution. Since $ABCD$ is a square of area 256 square inches, its sides must be $\sqrt{256} = 16$ inches long. Since triangle CEF is a right triangle of legs CE, CF, its area is $\frac{1}{2}|CE| \cdot |CF|$. So $\frac{1}{2}|CE| \cdot |CF| = 200$. But $CF = CE$. In fact, $\angle FCD = 90° - \angle FCB = \angle BCE$. Thus triangles CDF and CBE have an equal angle; since both are right triangles, they have all angles equal. Side CD is equal to side CB since $ABCD$ is a square. Thus triangles CDF and CBE are congruent and $CF = CE$. So $\frac{1}{2}|CE| \cdot |CF| = 200$ becomes $\frac{1}{2}|CE|^2 = 200$ or $|CE|^2 = 400$. By Pythagoras

$$|BE|^2 = |CE|^2 - |BC|^2 = 400 - 256 = 144.$$

It follows that $|BE| = \sqrt{144} = 12$ inches.

5. Achilles runs a certain distance. He runs the first half of the distance at a constant speed of 7 miles per hour. But then his heel is hurting him terribly, so he runs the second half at only 3 miles per hour. What was his average speed?

   (The average speed would be the speed that would have had to keep for the whole distance to finish in the same time)

   Solution. Since the distance is not important, let us suppose it is 42 miles. That is a bit long, but Achilles is strong and powerful. The point of 42 is that half of it is 21, which divides nicely by 7 and 3. At 7 miles per hour Achilles will take $21/7 = 3$ hours for the first half. It will take him $21/3 = 7$ hours for the second half. All in all, he runs the 42 miles in 10 hours, which means that his average speed was $42/10 = 4.2$ miles per hour.

6. In the following addition problem, **each letter stands for a different digit.**

	T	W	O
+	T	W	O
F	O	U	R

   If T = 7 and the letter O represents an even number, what is the only possible value for W?

   Solution. One first observation is that F = 1. It cannot be larger than 1 and since T = 7, it will be 1, not 0. Since T is 7 the second digit of the sum has to be either 4, if there is no carry from W+W, or 5, if there is a carry. But since that digit is O, so even, it has to be four and there is no carry from W+W. So we see O = 4, and W has to be one of 0, 1, 2, 3, 4; otherwise there is a carry. The last digit of the sum is O + O = 8, so R = 8 and there is no carry to the next digit. Of the four possibilities for W we can easily eliminate three. If W = 0, then U = 0, so U = W; but different letters represent different digits. We can't have W = 1 for the same reason, since F = 1, or W = 4, since O = 4. All that remains is W = 3, and that works. Thus W = 3.

7. A *Pythagorean triple* is a triple of positive integers a, b, c such that $a^2 + b^2 = c^2$. For example, $3, 4, 5$.

   (a) Show (explain why) 3, 4, 5 is the ONLY Pythagorean triple consisting of three consecutive integers.

   (b) Show (explain why) if a, b, c is a Pythagorean triple then so is $m \times a, m \times b, m \times c$, where m is any positive integer.

   (c) Find **ALL** Pythagorean triples a, b, c such that one of a or b is equal to 12.

   Solution. There are systematic ways of solving problems like these, but we'll proceed by brute force. The numbers are small.

   (a) Suppose a, b, c is a Pythagorean triple and $b = a+1, c = b+1 = a+2$. A fundamental formula for finding the square of a sum (known as a binomial) is

   $$(x + y)^2 = x^2 + 2xy + y^2.$$

   According to it

   $$c^2 = a^2 + b^2$$

   becomes (using $c^2 = (a + 2)^2 = a^2 + 4a + 4$, $b^2 = (a + 1)^2 = a^2 + 2a + 1$)

   $$a^2 + 4a + 4 = a^2 + 2a + 1 + a^2$$

   from which by some arithmetic we can get $a^2 - 2a = 3$ or

   $$a(a - 2) = 3.$$

   It follows that $a, a - 2$ must be divisors of 3 so one must be 1, the other one 3. Since $a - 2$ is the smaller one, we get $a - 2 = 1, a = 3$ thus $a = 3$. So $a = 3, b = 4, c = 5$. There is no other.

   (b) If $a^2 + b^2 = c^2$ then

   $$(ma)^2 + (mb)^2 = m^2a^2 + m^2b^2 = m^2(a^2 + b^2) = m^2c^2 = (mc)^2.$$

(c) Suppose $a = 12$. Then $144 + b^2 = c^2$ so that $c^2 - b^2 = 144$. If we use the fundamental equality

$$x^2 - y^2 = (x - y)(x + y)$$

we get

$$(c - b)(c + b) = 144.$$

The ways we can write 144 as a product of two positive integers, one strictly smaller than the other (this discards $144 = 12 \times 12$) are

$$144 = 1 \times 144 = 2 \times 72 = 3 \times 48 = 4 \times 36 = 6 \times 24 = 8 \times 18 = 9 \times 16.$$

The first integer factor should be $c - b$, the second factor $c + b$. Now $c - b, c + b$ have the same parity; both are either odd, or both are even. We can discard all products in which one factor is even, the other one odd. This leaves us with the following table

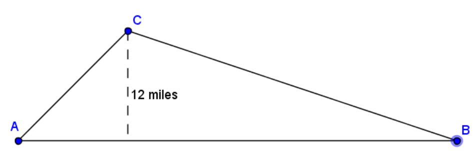
$c - b$	$c + b$	Solves to
2	72	$c = 37, b = 35$
4	36	$c = 20, b = 16$
6	24	$c = 15, b = 9$
8	18	$c = 13, b = 5$

So we have 4 such triples, namely

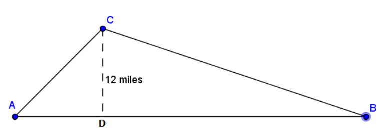
$$12, 5, 13; \qquad 12, 9, 15; \qquad 12, 16, 20; \quad 12, 35, 37.$$

To get those with $b = 12$ we just have to switch the first two entries in the triples we found.

8. To visit his beloved Krimhilde who lives at B, Siegfried, who lives at A, takes the straight route connecting A to B. But one day, bummer, the route from A to B is closed because the king's daughter lost one of her contact lenses while driving on it, and no one is allowed to use the road while all the king's men look for the lens. So Siegfried takes a different route, he takes a road going from A to C, then from C to B. The point C is exactly 12 miles away from the route AB, the distance from A to C is less than the distance from C to B. It turns out that by doing this detour, Siegfried traveled a total of 35 miles. **Knowing that the distances from A to C, from C to B and from A to B are all integers (in miles)**, what are all these distances? That is, how far is A from B, A from C, and C from B? Doing the previous problem first, can help.



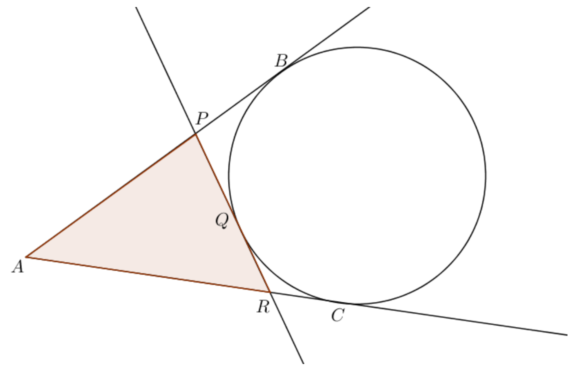
Solution. Suppose D is the point on route AB at least distance from C.



Then by the Pythagorean theorem $|AD|^2 + |DC|^2 = |AC|^2$. Since $|DC| = 12$ and $|AC|$ is an integer, $|AD|$ must also be an integer so that $12, |AD|, |AC|$ is a Pythagorean triple. For the same reason, $12, |DB|, |CB|$ is a Pythagorean triple. We also are given that $|AC| + |CB| = 35$. Looking at the list of triples we found in the previous problem, the only way we can get this last condition is if one of the triples is $12, 16, 20$, the other one $12, 9, 15$. Since $|AC| < |CB|$, the answer is

$$|AC| = 15, \qquad |CB| = 20, \qquad \text{and} \quad |AB| = |AD| + |DB| = 16 + 9 = 25.$$

9. From a point A outside of a circle we draw two tangents touching the circle at points B, C, respectively. We then draw a third tangent intersecting segment AB at P, segment AC at R and touching the circle at Q. If $|AB| = 20$, what is the perimeter of triangle APR? Can one even determine it from the provided data?



Solution. There is an honest solution and a cheating one. The cheating one has, however, a defect; it assumes there is a solution. First, the honest one. We use the fact that if we draw tangents from a point X outside of a circle and these tangents touch the circle at Y, Z, respectively, then $|XY| = |XZ$. Because of this result $|AB| = |AC|$, but also $|PQ| = |PB|$ and $|RQ| = |RC|$. If $\mathcal{P}$ is the perimeter of triangle APR then

$$\mathcal{P} = |AP|+|PR|+|RA| = |AP|+|PQ|+|QR|+|RA| = |AP|+|PB|+|RC|+|RA| = |AB|+|AC| = 2|AB| = 40.$$

The cheating solution assumes there is a solution that will be the same wherever Q is and moves Q in the direction of B. As this happens, side PR gets closer and closer to side AB, while side AR becomes negligible; in the end Q, R, B become indistinguishable and R collapses onto A. So in the limit we have a "triangle" with two sides equal to AB and one side equal to 0, so its perimeter is $2 \cdot 20 = 40$.